ON KULIKOV'S PROBLEM 1.67: THE INVARIANTS OF $N/[F,N]$ AT MINIMAL RANK

Abstract. Problem 1.67 of the Kourovka Notebook (L. Ya. Kulikov, 1965) asks for a complete system of invariants of $N/[F,N]$, where $F \twoheadrightarrow G$ is a presentation of a finitely presented group G on $d(G)$ generators with kernel N. By Hopf's formula $N/[F,N] \cong M(G) \oplus \mathbb{Z}^{d(G)-r_0}$, with $M(G) = H_2(G;\mathbb{Z})$ and r_0 the torsion-free rank of G_{ab}; so the free rank $d(G) - r_0 + \rho(M(G))$ together with the invariant factors of the torsion of $M(G)$ is a complete system, independent of the presentation. We determine which groups occur: $N/[F,N]$ surjects onto $\mathbb{Z}$ whenever G is not free, and every finitely generated abelian group of positive free rank occurs. The invariants are not computable from a finite presentation.

1. The problem and the results

Problem 1.67 of the Kourovka Notebook, posed by L. Ya. Kulikov in the first issue (1965), reads as follows [7, Problem 1.67].

> **1.67.** Suppose that G is a finitely presented group, F a free group whose rank is equal to the minimal number of generators of G, with a fixed homomorphism of F onto G with kernel N. Find a complete system of invariants of the factor-group of N by the commutator subgroup $[F,N]$.
>
> *L. Ya. Kulikov*

Notation. Throughout, G is finitely presented, $d = d(G)$ is its minimal number of generators, F is free on $x_1,\dots,x_d$, $\pi\colon F \twoheadrightarrow G$ is an epimorphism and $N = \ker\pi$. We write

$$Q = Q_\pi = N/[F,N], \qquad E = E_\pi = F/[F,N],$$

so that Q is central in E and $1 \to Q \to E \to G \to 1$ is a central extension. Further, $r_0 = r_0(G_{\mathrm{ab}})$ is the torsion-free rank of $G_{\mathrm{ab}} = G/G'$, and $M(G) = H_2(G;\mathbb{Z})$ is the Schur multiplier. For a finitely generated abelian group A, $\rho(A)$ is its free rank and $\mathrm{T}(A)$ its torsion subgroup; γ_k denotes the lower central series.

The problem asks for invariants of the *factor-group* Q — the Russian text is, if anything, more explicit on this point[1] — and that is the question answered by Theorems 1.1 and 1.2. Whether the invariants can be *computed* from a finite presentation is a different question, settled negatively by a standard Adian–Rabin argument (Remark 3.3). Finally, Kulikov fixes the homomorphism π; the group Q does not depend on it at all (Theorem 1.1), and Remark 3.4 records what does.

Theorem 1.1. *$Q \cong M(G) \oplus \mathbb{Z}^{\,d(G)-r_0}$, and the isomorphism type of Q depends only on G — not on the choice of minimal generating tuple nor of π. Consequently a complete system of invariants of Q is the pair*

$$I(G) = \big(\, d(G) - r_0(G_{\mathrm{ab}}) + \rho(M(G))\,;\ e_1 \mid e_2 \mid \cdots \mid e_s \,\big),$$

where $e_1 \mid \cdots \mid e_s$ are the invariant factors of $\mathrm{T}(M(G))$: for finitely presented G_1, G_2 one has $Q(G_1) \cong Q(G_2)$ if and only if $I(G_1) = I(G_2)$.

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[1] The Russian original [9, 1.67] asks for a complete system of invariants "of the factor-group" (*faktor-gruppy*) of N by the mutual commutator subgroup $[F,N]$; the homomorphism is part of the given data.

Theorem 1.2. (a) *If G is not free, then Q surjects onto $\mathbb{Z}$; hence $Q = 0$ if and only if G is free, and Q is never finite and nontrivial.* (b) *The isomorphism types of Q, as G ranges over all finitely presented groups, are exactly $\{0\} \cup \{A \text{ f.g. abelian} : \rho(A) \geq 1\}$.*

Classical and new. Theorem 1.1 is Hopf's formula [6] plus a splitting, and appears verbatim as Proposition 1 of Hartung [5]; we include the short proof. Theorem 1.2(a) sharpens Stallings' freeness criterion [15] (an n-generated group with $H_1 \cong \mathbb{Z}^n$ and $H_2 = 0$ is free) by locating a $\mathbb{Z}$-quotient of $N/[F,N]$ for every non-free G; part (b) and the realization question appear to be unposed in print. The uncomputability of $I(G)$ (Remark 3.3) is an observation riding on the Adian–Rabin theorem. This note was produced by an automated pipeline built on a large language model and verified independently of its authorship; see the final section.

2. The abelian invariant

Lemma 2.1. *N is the normal closure in F of finitely many elements $r_1, \dots, r_m$, and Q is central in E, abelian, and generated by the images $\bar{r}_1, \dots, \bar{r}_m$; in particular Q is finitely generated.*

Proof. G is finitely presented and $x_1\pi, \dots, x_d\pi$ is a finite generating set, so G is finitely presented on it (finite presentability does not depend on the finite generating set, B. H. Neumann [11]): $N = \langle\langle r_1, \dots, r_m \rangle\rangle$. Centrality of Q in E is the definition of $[F,N]$; Q is abelian since $[N,N] \leq [F,N]$. Any $n \in N$ is a product of conjugates $(r_j^{\pm 1})^w$, and modulo $[F,N]$ each conjugate equals $r_j^{\pm 1}$. □

We use Hopf's formula [6] as known: for every free presentation $G = F/N$,

$$(N \cap F')/[F,N] \cong M(G). \tag{1}$$

Proof of Theorem 1.1. Since $[F,N] \leq N \cap F'$, the map $N \to NF'/F'$ induces an exact sequence

$$0 \to (N \cap F')/[F,N] \to Q \to NF'/F' \to 0.$$

As $F/NF' \cong G_{\mathrm{ab}}$, the subgroup $NF'/F' = \ker(F_{\mathrm{ab}} \to G_{\mathrm{ab}})$ of $F_{\mathrm{ab}} \cong \mathbb{Z}^d$ is free abelian of rank $d - r_0$ (tensor $0 \to NF'/F' \to F_{\mathrm{ab}} \to G_{\mathrm{ab}} \to 0$ with $\mathbb{Q}$). A free abelian quotient lifts, so the sequence splits and $Q \cong (N \cap F')/[F,N] \oplus \mathbb{Z}^{d-r_0} \cong M(G) \oplus \mathbb{Z}^{d-r_0}$ by (1). Both summands are isomorphism invariants of G, as $d = d(G)$ is fixed by minimality. Hence $\mathrm{T}(Q) \cong \mathrm{T}(M(G))$ and $\rho(Q) = d - r_0 + \rho(M(G))$, and the fundamental theorem of finitely generated abelian groups (free rank and invariant factors form a complete system) gives the rest. □

Remark 2.2. Minimality is not used in the splitting: a free presentation of any finite rank $k \geq d$ gives $Q \cong M(G) \oplus \mathbb{Z}^{k-r_0}$ [5, Prop. 1]. Kulikov's normalization $\operatorname{rk} F = d(G)$ is exactly what makes Q a well-defined invariant of G. Also, since G is finitely presented, $M(G)$ is finitely generated (Lemma 2.1 and Theorem 1.1).

Remark 2.3 (The formula is not effective). Theorem 1.1 reduces the invariants of Q to $d(G)$, $r_0(G_{\mathrm{ab}})$ and $M(G) = H_2(G;\mathbb{Z})$, but H_2 cannot be "unfolded" from a finite presentation: the set of finite presentations defining a group with $H_2 = 0$ is not recursive (Gordon; González-Acuña–Gordon–Simon [2, Cor. 5.5]), so no algorithm decides from a presentation whether $\mathrm{T}(Q) = 0$ or computes $\rho(Q)$. Remark 3.3 gives a second, independent route to the same conclusion.

Example 2.4. For $G = \mathbb{Z}/m$, $m \geq 2$: $d = 1$, $F = \langle x \rangle$, $N = \langle x^m \rangle$, $[F,N] = 1$, so $Q \cong \mathbb{Z}$ for every m — Q forgets almost all of G (in accordance with $M(\mathbb{Z}/m) = 0$, $d - r_0 = 1$). Similarly $Q(\mathbb{Z}/2 \times \mathbb{Z}/2) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$ and $Q(\mathbb{Z}^k) \cong \Lambda^2\mathbb{Z}^k$.

3. Which groups occur

Proof of Theorem 1.2(a). First, $N = 1$ if and only if G is free: $N = 1$ means $G \cong F_d$; conversely, if G is free its rank is $d(G)$, and then π is an epimorphism between free groups of the same finite rank, hence injective, free groups being Hopfian. Now let $N \neq 1$. Since F is residually nilpotent, $\bigcap_c \gamma_c(F) = 1$ (Magnus [8]), there is a largest $k \geq 1$ with $N \leq \gamma_k(F)$. Then $[F, N] \leq \gamma_{k+1}(F)$, so Q surjects onto $N\gamma_{k+1}(F)/\gamma_{k+1}(F) \leq \gamma_k(F)/\gamma_{k+1}(F)$, which is nontrivial by the maximality of k and free abelian (Magnus–Witt [19]). A nontrivial free abelian quotient of a finitely generated abelian group forces $\rho(Q) \geq 1$. Consequently $Q = 0$ iff G is free, and a finite Q has $\rho(Q) = 0$, hence G free, hence $Q = 0$. □

Corollary 3.1. *If G is not free and $G_{\mathrm{ab}} \cong \mathbb{Z}^{d(G)}$, then $M(G)$ has free rank at least 1; in particular $M(G)$ is infinite.*

Proof. Here $d = r_0$, so $\rho(Q) = \rho(M(G))$ by Theorem 1.1, and $\rho(Q) \geq 1$ by Theorem 1.2(a). □

Stallings' criterion [15] says $M(G) \neq 0$ under these hypotheses; Corollary 3.1 says the free rank is positive. (Consistent with surface groups, $M = \mathbb{Z}$; free nilpotent groups, M free abelian; right-angled Artin groups, $M = \mathbb{Z}^{\#\mathrm{edges}}$.)

Lemma 3.2. *For finitely presented A_1, A_2: $Q(A_1 * A_2) \cong Q(A_1) \oplus Q(A_2)$.*

Proof. $d(A_1 * A_2) = d(A_1) + d(A_2)$ (Grushko [4]); r_0 is additive, as $(A_1 * A_2)_{\mathrm{ab}} = A_{1,\mathrm{ab}} \oplus A_{2,\mathrm{ab}}$; and $M(A_1 * A_2) \cong M(A_1) \oplus M(A_2)$, since $K(A_1 * A_2, 1) = K(A_1, 1) \vee K(A_2, 1)$ and Mayer–Vietoris applies. Now use Theorem 1.1. □

Proof of Theorem 1.2(b). Necessity is part (a), with $Q = 0$ for free G. For sufficiency let $A = \mathbb{Z}^n \oplus \mathbb{Z}/e_1 \oplus \cdots \oplus \mathbb{Z}/e_s$ with $n \geq 1$, $e_i \geq 2$. By Example 2.4 and Lemma 3.2, $Q((\mathbb{Z}/2)^{*n}) \cong \mathbb{Z}^n$, which settles $s = 0$. Let $s \geq 1$.

Torsion block. Choose primes $q_1, \ldots, q_s$ with $q_i \equiv 1 \pmod{e_i}$ and $q_i > 9$ such that the groups $S_i := \mathrm{PSL}(e_i, q_i)$ have pairwise distinct orders, hence are pairwise non-isomorphic: Dirichlet's theorem supplies infinitely many admissible q_i for each i, and along primes $q \equiv 1 \pmod{e_i}$ the order $|\mathrm{PSL}(e_i, q)| = q^{e_i(e_i-1)/2} \prod_{j=2}^{e_i} (q^j - 1)/e_i$ is strictly increasing in q, so the q_i can be chosen in turn avoiding the finitely many orders already used. Each S_i is 2-generated (Steinberg [17]), and $M(S_i) \cong \mathbb{Z}/\gcd(e_i, q_i - 1) = \mathbb{Z}/e_i$ (Steinberg [16]; the exceptional multipliers of $\mathrm{PSL}(n, q)$ all have $q \leq 9$ [3]). Put $H := S_1 \times \cdots \times S_s$ and $G_1 := H \times \mathbb{Z}$.

(i) $M(G_1) \cong \mathbb{Z}/e_1 \oplus \cdots \oplus \mathbb{Z}/e_s$: by the Künneth formula for $K(\cdot, 1)$'s, $M(A \times B) \cong M(A) \oplus M(B) \oplus (A_{\mathrm{ab}} \otimes B_{\mathrm{ab}})$, and all cross terms vanish because the S_i and H are perfect and $M(\mathbb{Z}) = 0$.

(ii) $(G_1)_{\mathrm{ab}} \cong \mathbb{Z}$, so $r_0 = 1$.

(iii) $d(G_1) = 2$. As G_1 is nonabelian, $d \geq 2$. Pick $a = (a_1, \ldots, a_s)$, $b = (b_1, \ldots, b_s)$ with $\langle a_i, b_i \rangle = S_i$ (so $a_i \neq 1$). Then $\langle a, b \rangle$ projects onto every factor of H, and a subgroup K of a product of pairwise non-isomorphic nonabelian simple groups projecting onto every factor is the whole product: by induction on s, with $H = H' \times S_s$, the subgroup $K \cap (1 \times S_s)$ is normal in S_s (it is normalized by K, which projects onto S_s), so it is S_s — and then $K \supseteq 1 \times S_s$ covers H', so $K = H$ — or 1, in which case $K \cong H'$ maps onto S_s; but normal subgroups of a product of nonabelian simple groups are subproducts, so $S_s \cong S_i$ for some $i < s$, a contradiction. Hence $H = \langle a, b \rangle$, and $G_1 = \langle (a, 0), (b, 1) \rangle$: this subgroup projects onto H, so it contains $\langle\langle a \rangle\rangle_H \times 0 = H \times 0$ (each $a_i \neq 1$ and S_i is simple), hence also $(1, 1)$.

By Theorem 1.1, $Q(G_1) \cong M(G_1) \oplus \mathbb{Z}^{2-1} \cong \mathbb{Z}/e_1 \oplus \cdots \oplus \mathbb{Z}/e_s \oplus \mathbb{Z}$, and $G := G_1 * (\mathbb{Z}/2)^{*(n-1)}$ gives $Q(G) \cong A$ by Lemma 3.2. All witnesses are finitely presented. □

Remark 3.3 (Non-effectivity). No algorithm computes $I(G)$ from a finite presentation of G — not even the single number $\rho(Q)$. Indeed $\rho(Q) = 0$ iff G is free (Theorem 1.2(a)), and freeness

is a Markov property (free groups are finitely presented; $\mathbb{Z}/2$ embeds in no free group), hence undecidable by the Adian–Rabin theorem [13], [10, Thm. 3.3, Cor. 3.4]. The ingredient $d(G)$ is likewise uncomputable (deciding $d = 0$ is deciding triviality). Independently of the freeness reduction, the $M(G)$-summand alone is non-effective (Remark 2.3). The nilpotent quotients of E are computable and approximate $M(G)$ in favourable cases [5]; no such process can terminate with a certificate in general.

Remark 3.4 (The extension itself). Kulikov's clause "with a fixed homomorphism" invites the question one level up: is the central extension $E_\pi = F/[F, N]$ of G by Q determined by G? Not in general. For the triangle groups $\Delta(p, p, p)$ with $p \equiv 1 \pmod 4$ the minimal generating pairs (a, b) and (a, b^j), j a quadratic non-residue modulo p, give non-isomorphic groups $F/[F, N]$; on the other hand E_π is determined by G and $\operatorname{rk} F$ whenever G_{ab} is torsion-free or G is abelian. The relation module $N/[N, N]$ varies with π already at minimal rank (Dunwoody [1]), so the dependence dies exactly at the coinvariants Q. The proofs will appear elsewhere.

Methods and AI disclosure

This note was produced by an automated research pipeline built on a large language model (`claude-fable-5`), and its claims were verified before assembly: the claim package of Sections 2–3 passed two independent adversarial verification passes by fresh-context instances shown only the problem statement and the candidate text, in which every lemma was re-derived and every computation independently reprogrammed from the stated data, together with a citation audit checking every reference against its source. Every computational claim was reproduced on independent implementations: the identity $Q \cong M(G) \oplus \mathbb{Z}^{d-r_0}$ on explicit minimal presentations (including two distinct minimal presentations each of S_3 and of the trefoil group), in GAP 4.15.1 [18] with the packages FGA 1.5.0 [14] and nq 2.5.11 [12], and in independently written pure-Python programs using no group-theory library. The results announced in Remark 3.4 were established and verified by the same process. Complete machine records — scripts, transcripts and verifier reports — are preserved in the repository accompanying this note and are available from the authors.

To our knowledge, Theorem 1.2 is new here; Theorem 1.1 and Remarks 2.3 and 3.3 are classical in substance and are included for completeness, and everything else is cited where used. No published treatment of Problem 1.67 was found in our searches, with the caveat that zbMATH and MathSciNet were not consulted.

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