MIXED VARIETIES OF FINITE p-GROUPS OF UNBOUNDED NILPOTENCY CLASS

ABSTRACT. For every prime p and every integer $k \geq 2$, we construct a finite p-group of nilpotency class k whose generated mixed variety is not an ordinary variety. A mixed power–commutator identity excludes a cyclic subgroup from the mixed variety. The construction extends Anashin's odd-prime class-two example and gives a negative answer to Kourovka Problem 10.4 for each prime, including 2.

1. The problem and the construction

Problem 10.4 of the *Kourovka Notebook* [2] reads:

> Let p be a prime number. Is it true that a mixed variety of groups generated by an arbitrary finite p-group of sufficiently large nilpotency class is a variety of groups?
> V. S. Anashin

We use Anashin's definition [1, p. 172]. For a group G, write $G[X] = G * F(X)$, where $F(X)$ is the free group on a countable set X of variables. Let $I_G[X]$ be the set of *mixed identities*: the words in $G[X]$ that evaluate to 1 under every substitution from G. A group H belongs to $\mathrm{mvar}(G)$ if there is a homomorphism $\phi : G \to H$ such that its extension $\widehat{\phi} : G[X] \to H[X]$, fixing the variables, satisfies

$$\widehat{\phi}(I_G[X]) \subseteq I_H[X].$$

The coefficient homomorphism is not required to be injective, surjective or nontrivial. In particular $G \in \mathrm{mvar}(G)$. We use $[x, y] = x^{-1}y^{-1}xy$.

Theorem 1. *Fix a prime p, put $s = 1$ for odd p and $s = 2$ for $p = 2$, and let $n \geq s + 1$. The group*

$$G_{p,n} = \left\langle a, b \;\middle|\; a^{p^{n-s}} = b^{p^n} = 1, \quad a^{-1}ba = b^{1+p^s} \right\rangle \tag{1}$$

is a finite p-group of order p^{2n-s} and nilpotency class $\lceil n/s \rceil$. Its mixed variety is not an ordinary group variety. Consequently, for every prime p and every $k \geq 2$, such a group of class k exists.

Anashin's class-two example for odd p [1, p. 176] is the case $n = 2$ of (1). The definition of mixed variety and that starting example are classical inputs. The contribution under review is the extension to arbitrary n, the two-primary construction and the resulting unbounded-class obstruction. All congruences and group calculations needed for this extension are proved below. No exact prior full solution was located in the recorded literature search; that search does not certify originality.

2. A mixed-identity obstruction

Suppose a group G satisfies a mixed identity

$$x^N = [x, c] \qquad (c \in G), \tag{2}$$

and has an abelian subgroup B containing an element whose order does not divide N. For every coefficient homomorphism $\phi : G \to B$, the transported word $x^N[x, \phi(c)]^{-1}$ evaluates at $y \in B$ as y^N. It is nontrivial for some $y \in B$, so no coefficient map transports all mixed identities and

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$B \notin \mathrm{mvar}(G)$. Since $G \in \mathrm{mvar}(G)$, this class is not closed under subgroups. Every ordinary identity is inherited by subgroups; hence $\mathrm{mvar}(G)$ cannot be an ordinary variety.

3. Two congruences

Retain p, s, n from Theorem 1 and put $q = 1 + p^s$, $N = p^{n-1}$.

Lemma 2. *For every integer $r \geq 0$,*

$$q^{p^r} \equiv 1 + p^{s+r} \pmod{p^{s+r+1}}. \tag{3}$$

For every integer $u \equiv 1 \pmod{p^s}$,

$$S_N(u) := 1 + u + \cdots + u^{N-1} \equiv N \pmod{p^n}. \tag{4}$$

Proof. If $e \geq s$ and v is prime to p, the binomial expansion gives

$$(1 + vp^e)^p \equiv 1 + vp^{e+1} \pmod{p^{e+2}}.$$

For odd p, the intermediate terms have p-adic valuation at least $1 + 2e \geq e + 2$, and the last term has valuation $ep \geq e + 2$. For $p = 2$, the remaining quadratic term has valuation $2e \geq e + 2$. Starting from $q = 1 + p^s$ and applying this congruence inductively preserves the leading coefficient 1 modulo p, proving (3).

For (4), set $z = u - 1$ and expand the polynomial identity

$$S_N(1 + z) = \sum_{h=0}^{N-1} \binom{N}{h+1} z^h.$$

If $z = 0$ the assertion is immediate. Otherwise, for $h \geq 1$, the identity $(h + 1)\binom{N}{h+1} = N\binom{N-1}{h}$ implies

$$v_p\left(\binom{N}{h+1} z^h\right) \geq n - 1 - v_p(h + 1) + sh \geq n.$$

To verify the last inequality, for odd p one has $v_p(h + 1) \leq h - 1$, since the contrary would imply $3^h \leq p^h \leq h + 1$. For $p = 2$ one has $v_2(h + 1) \leq h$, since $2^{h+1} > h + 1$, and therefore $2h - v_2(h + 1) \geq h \geq 1$. Thus every nonconstant term vanishes modulo p^n, while the constant term is N. □

4. The mixed law and the nilpotency class

By (3), multiplication by q on the cyclic group of order p^n has exact order p^{n-s}: its p^{n-s}-th power is trivial, whereas its p^{n-s-1}-th power is not. Thus (1) presents the corresponding semidirect product of two cyclic p-groups. More explicitly, its relations collect every word into $a^i b^j$, with $0 \leq i < p^{n-s}$ and $0 \leq j < p^n$; the semidirect product realizes all these normal forms distinctly. In particular, $|G_{p,n}| = p^{2n-s}$ and $B = \langle b \rangle$ has order p^n.

The oriented normal-form multiplication is

$$(a^i b^j)(a^{i'} b^{j'}) = a^{i+i'} b^{jq^{i'}+j'}.$$

Since $p^{n-s} \mid N$ and $q^i \equiv 1 \pmod{p^s}$, (4) gives

$$(a^i b^j)^N = a^{iN} b^{jS_N(q^i)} = b^{jN}. \tag{5}$$

Put $m = p^{n-s-1}$. Equation (3) gives $q^m \equiv 1 + N \pmod{p^n}$, whence

$$[a^i b^j, a^m] = b^{-j} a^{-m} b^j a^m = b^{j(q^m-1)} = b^{jN}.$$

Together with (5), this proves the mixed identity

$$x^N = [x, a^m]$$

on $G_{p,n}$. But b has order $p^n = pN$, so the abelian subgroup B fails $x^N = 1$. The obstruction (2) proves that $\mathrm{mvar}(G_{p,n})$ is not an ordinary variety.

For completeness, its lower central series is

$$\gamma_{r+1}(G_{p,n}) = \langle b^{p^{sr}} \rangle \qquad (r \geq 1). \tag{6}$$

Indeed, the quotient by $\langle b^{p^s} \rangle$ is abelian, and $[b, a] = b^{p^s}$ generates that subgroup. For the induction step, B centralizes each $\langle b^{p^{sr}} \rangle$, whereas

$$[b^{p^{sr}}, a] = b^{p^{s(r+1)}}.$$

Every commutator of an element of $\langle b^{p^{sr}} \rangle$ with an arbitrary $a^i b^j$ belongs to $\langle b^{p^{s(r+1)}} \rangle$, since $p^s \mid q^i - 1$. This proves both inclusions in the recurrence and hence (6). Its right-hand side is trivial exactly when $sr \geq n$, so the class is $\lceil n/s \rceil$. For any $k \geq 2$, choosing $n = k$ for odd p and $n = 2k$ for $p = 2$ gives class k, completing the proof of Theorem 1.

Open questions

The examples rule out a class threshold even when it may depend on the fixed prime. They do not classify the finite p-groups whose mixed varieties are ordinary varieties.

Methods and AI disclosure

This draft was assembled with `gpt-6-astra` in Codex from an internal proof candidate. The original argument passed two independent adversarial reviews; a separate citation audit checked Anashin's definition against the retained primary page. Independent Python and GAP calculations corroborated finite instances of the mixed law and lower central series. No computation is a premise of the proof. Source notes, reports and transcripts are preserved in the accompanying repository under `problems/10.4/`. This is an internal candidate awaiting human assessment, without a claim of publication, editor acceptance or certified originality.

References

1. V. S. Anashin, *Mixed identities and mixed varieties of groups*, Math. USSR-Sb. **57** (1987), no. 1, 171–182.
2. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory: The Kourovka notebook*, 21 ed., Sobolev Institute of Mathematics, Novosibirsk, 2026, Version arXiv:1401.0300v45, 3 July 2026, Problem 10.4.