ADJOINT HOMOMORPHISMS OUTSIDE THE WEYL-ELEMENT CONDITION IN KOUROVKA PROBLEM 10.43

Abstract. For every nonzero commutative ring K with 2 invertible and every even $n \geq 4$, conjugation on $M_n(K)$ gives an injective homomorphism $PE_n(K) \to \mathrm{GL}_{n^2}(K)$ which fails the simultaneous Weyl-element condition in Kourovka Problem 10.43. A central relation excludes the condition even when its embedding is an arbitrary injective group homomorphism. This supplies the particular cases permitted by the question.

1. The problem and the result

Problem 10.43, posed by V. M. Petechuk, reads as follows [1, p. 41].

> Let R and S be associative rings with identity such that 2 is invertible in S. Let $\Lambda_I : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)$ be the homomorphism corresponding to an ideal I of R and let $E_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by elementary transvections $t_{ij}(x)$. Let $a_{ij} = t_{ij}(1)t_{ji}(-1)t_{ij}(1)$, let the bar denote images in the factor-group of $\mathrm{GL}_n(R)$ by the centre and let $PG = \overline{G}$ for $G \leq \mathrm{GL}_n(R)$. A homomorphism
>
> $$\Lambda : E_n(R) \to \mathrm{GL}(W) = \mathrm{GL}_m(S)$$
>
> is called standard if $S^m = P \oplus \cdots \oplus P \oplus Q$ (a direct sum of S-modules in which there are n summands P) and
>
> $$\Lambda x = g^{-1}\tau\big(\delta^*(x)f + ({}^t\delta^*(x)^\nu)^{-1}(1-f)\big)g, \qquad x \in E_n(R),$$
>
> where $\delta^* : \mathrm{GL}_n(R) \to \mathrm{GL}_n(\operatorname{End} P)$ is the homomorphism induced by a ring homomorphism $\delta : R \to \operatorname{End} P$ taking identity to identity, g is an isomorphism of the module W onto S^m, $\tau : \mathrm{GL}_n(\operatorname{End} P) \to \mathrm{GL}(gW)$ is an embedding, f is a central idempotent of δR, t denotes transposition and ν is an antiisomorphism of δR. Let $n \geq 3$, $m \geq 2$. One can show that the homomorphism
>
> $$\Lambda_0 : PE_n(R) \to \mathrm{GL}(W) = \mathrm{GL}_m(S)$$
>
> is induced by a standard homomorphism Λ if $\Lambda_0\overline{a}_{ij} = g^{-1}\tau(a^*_{ij})g$ for some g and τ and for any $i \neq j$ where a^*_{ij} denotes the matrix obtained from a_{ij} by replacing 0 and 1 from R by 0 and 1 from $\operatorname{End} P$. Find (at least in particular cases) the form of a homomorphism Λ_0 that does not satisfy the last condition.

Thus $PE_n(R)$ is the image of $E_n(R)$ modulo $\mathrm{Z}(\mathrm{GL}_n(R))$. The last condition requires one pair (g, τ) to work simultaneously for all $i \neq j$. We allow τ to be any injective group homomorphism of the indicated type.

Theorem 1. *Let K be a nonzero commutative ring with identity in which 2 is invertible, and let $n \geq 4$ be even. With $W = M_n(K)$ and $m = n^2$, the map*

$$\Lambda_0 : PE_n(K) \longrightarrow \mathrm{GL}(W), \qquad \Lambda_0(\overline{x})(X) = xXx^{-1},$$

is a well-defined injective homomorphism and does not satisfy the last condition in Problem 10.43. In the basis (E_{ij}) of matrix units its entries are

$$\Lambda_0(\overline{x})_{(i,j),(k,l)} = x_{ik}(x^{-1})_{lj}.$$

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In particular, $K = \mathbb{Q}$ and $n = 4$ give an example of degree 16. The theorem gives particular cases, not a classification of homomorphisms for arbitrary rings and degrees. Conjugation is a classical construction; all its required properties are proved below. The deduction specific to the problem is the obstruction supplied by a central signed-matrix relation. The sufficient-standardness assertion quoted in the problem is not used.

2. Proof

Proof of Theorem 1. First, the centre of $\mathrm{GL}_n(K)$ consists of the invertible scalar matrices. Indeed, a central matrix C commutes with $1_n + E_{ij}$, hence with E_{ij}, for all $i \neq j$. Comparing entries in $CE_{ij} = E_{ij}C$ shows that all off-diagonal entries of column i vanish and that $C_{ii} = C_{jj}$. Thus $C = c1_n$, and invertibility makes c a unit. The converse follows from commutativity of K.

Conjugation by x is K-linear and invertible, and conjugation by xy is the composite of conjugation by x and by y. Multiplying x by an invertible scalar does not change this action, so it descends to $PE_n(K)$. If the action fixes all of $M_n(K)$, then x commutes with all matrix units and the preceding calculation makes x scalar. This proves injectivity. The entry formula follows by applying the action to E_{kl}.

Put $w_r = a_{2r-1,2r}$ for $1 \leq r \leq n/2$. The nontrivial block of a_{ij} on coordinates i, j is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Its square is -1 on those two coordinates and $+1$ elsewhere. As these pairs partition the coordinates,

$$\prod_{r=1}^{n/2} w_r^2 = -1_n, \qquad \prod_{r=1}^{n/2} \overline{w}_r^2 = 1 \quad \text{in } PE_n(K). \tag{1}$$

Suppose that the last condition holds for a decomposition $K^{n^2} = P^n \oplus Q$, an isomorphism g, and an embedding

$$\tau : \mathrm{GL}_n(\mathrm{End}_K P) \longrightarrow \mathrm{GL}(gW).$$

Assume first that $P \neq 0$. Multiplication by 2 is invertible on P, so $-\mathrm{id}_P \neq \mathrm{id}_P$. The same block multiplication over $\mathrm{End}_K P$ gives

$$\prod_{r=1}^{n/2} (w_r^*)^2 = -\mathrm{id}_{P^n} \neq \mathrm{id}_{P^n}.$$

Applying Λ_0 to (1) and substituting the simultaneous identities therefore yields

$$\mathrm{id}_W = g^{-1}\tau(-\mathrm{id}_{P^n})g,$$

contrary to injectivity of τ.

If $P = 0$, then $\mathrm{GL}_n(\mathrm{End}_K P)$ is trivial and every $\tau(a_{ij}^*)$ is the identity. But conjugation by a_{12} sends E_{11} to E_{22}, which is different from E_{11} because $K \neq 0$. This also contradicts the condition. □

Open questions

The form of homomorphisms outside the condition for arbitrary R, S, n, m remains beyond the present result. In particular, the central relation used here does not provide an answer for odd n.

Methods and AI disclosure

This internal candidate was developed and typeset in Codex using `gpt-6-astra`. Before typesetting, the argument passed two independent adversarial reviews, each supplied with the problem and candidate argument rather than the other review. A source audit checked the original statement and confirmed that no external theorem is needed. The matrix identities were corroborated independently in Python and GAP for $n = 4, 6, 8$; these checks are not premises of the symbolic proof. Searches located no exact prior answer, without establishing priority. The adjoint representation itself is classical. Source files, review reports, citation records, scripts and transcripts are retained in the accompanying Kourovka repository under `problems/10.43/`. Human mathematical acceptance remains pending.

References

1. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory. The Kourovka Notebook*, 2026, arXiv:1401.0300v45, `https://arxiv.org/abs/1401.0300v45`.