A CHARACTERISTIC-TWO COUNTEREXAMPLE TO KOUROVKA NOTEBOOK 10.46

Abstract. An explicit finitely presented noncommutative algebra R over $\mathbb{F}_2$ admits a matrix $\sigma \in \mathrm{GL}_3(R)$ for which $[[\sigma, T_1], T_2]$ has infinite order for every pair of nonidentity matrices satisfying $(T_i - I)^2 = 0$. Thus no such double commutator is unipotent. The proof uses Gerasimov's structure theorem for the units of a coproduct of rings and an elementary free-product tree argument.

1. Statement and conventions

Problem 10.46 of the Kourovka Notebook, posed by V. M. Petechuk, asks:

> Prove that for every element $\sigma \in \mathrm{GL}_n(R)$, $n \geq 3$, there exist transvections τ_1, τ_2 such that $[[\sigma, \tau_1], \tau_2]$ is a unipotent element.

The source is arXiv:1401.0300v45, printed page 41. The local setup at 10.43 uses associative rings R, S with identity and requires 2 to be invertible in S, not in R. Problem 10.46 does not separately require R to be commutative. We address that literal scope, not a commutative-ring or 2-invertible variant.

All rings below are associative and unital. Set $[a, b] = a^{-1}b^{-1}ab$. A matrix U is *unipotent* if $U - I$ is nilpotent. Transvections are understood to be nonidentity. In fact, we cover every $T \neq I$ satisfying $(T - I)^2 = 0$, hence all usual narrower transvection classes. Allowing the identity as a transvection would make the question vacuous.

2. The explicit ring and matrix

Let $k = \mathbb{F}_2$. Define the k-algebra R by eighteen noncommuting generators x_{ij}, y_{ij}, $1 \leq i, j \leq 3$, subject to

$$\sum_{\ell=1}^{3} x_{i\ell} y_{\ell j} = \delta_{ij}, \qquad \sum_{\ell=1}^{3} y_{i\ell} x_{\ell j} = \delta_{ij} \quad (1 \leq i, j \leq 3). \tag{1}$$

Write $X = (x_{ij})$, $Y = (y_{ij})$, and $H = I + E_{12}$, where E_{ij} are the standard matrix units over R. Take

$$\sigma = HX = \begin{pmatrix} x_{11} + x_{21} & x_{12} + x_{22} & x_{13} + x_{23} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \qquad \sigma^{-1} = YH. \tag{2}$$

Indeed $XY = YX = I$ and $H^2 = I$. The assignment $x_{ij} = y_{ij} = \delta_{ij}$ defines a unital homomorphism $R \to k$. Thus $R \neq 0$, its characteristic is exactly two, and $E_{12} \neq 0$. It is genuinely noncommutative: over the free algebra $k\langle \alpha, \beta \rangle$, both X and Y may be specialized to $I + \alpha E_{12} + \beta E_{13}$, whose square is I. This gives a surjection $R \to k\langle \alpha, \beta \rangle$.

Theorem 1. *For this ring and matrix, if $T_1, T_2 \in \mathrm{GL}_3(R)$ are nonidentity and $(T_i - I)^2 = 0$, then $[[\sigma, T_1], T_2]$ has infinite order. In particular it is not unipotent.*

3. The unit group of a coproduct

We use one external structure theorem. In the field-algebra specialization of Gerasimov's Definition 1.6 and Main Theorem 1.10 [1, p. 42], let $C = A *_k B$ be the coproduct of directly finite unital k-algebras. Here *directly finite* means that $\xi\eta = 1$ implies $\eta\xi = 1$. Define $G(A, C)$ to be the subgroup of $\mathrm{U}(C)$ generated by $\mathrm{U}(A)$ and the elements $1 + \eta\gamma\xi$, where $\xi, \eta \in A$, $\xi\eta = 0$,

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and γ is a product of elements of the two algebra factors. Define $G(B, C)$ similarly. The theorem gives the internal amalgamated free product

$$\mathrm{U}(C) = G(A, C) *_{k^\times} G(B, C). \tag{3}$$

The factors need not be domains or commutative; direct finiteness is the relevant hypothesis.

Apply this with

$$A = M_3(k), \qquad B = k[t, t^{-1}], \qquad C = A *_k B.$$

The algebra A is directly finite: a right inverse of a linear map on a finite-dimensional vector space is also a left inverse. The algebra B is a commutative domain, so is directly finite. Both contain the same scalar field with the same identity. Since B has no zero divisors, its additional generators $1 + \eta\gamma\xi$ with $\xi\eta = 0$ are all 1. Comparing the lowest and highest Laurent exponents shows that its units are precisely t^m, $m \in \mathbb{Z}$, because $k^\times = \{1\}$. Therefore

$$\mathrm{U}(C) = P * \langle t \rangle, \qquad P = G(A, C), \tag{4}$$

where $\langle t \rangle$ is infinite cyclic. This is an internal ordinary free product of subgroups of the entire unit group.

Write e_{ij} for the matrix units in $A \subseteq C$. In particular $h = 1 + e_{12}$ is a nonidentity element of P. The canonical factor embeddings can also be checked by the map $C \to M_3(k[z, z^{-1}])$ which includes A in the standard way and sends t to zI. Its restrictions to both factors are injective, and their images intersect in the constant scalar matrices.

4. Identification with the presented ring

Let $D = e_{11}Ce_{11}$, with identity e_{11}. Matrix-unit identities give mutually inverse algebra isomorphisms

$$\begin{aligned} &\Phi : C \longrightarrow M_3(D), && \Phi(\gamma)_{ij} = e_{1i}\gamma e_{j1}, \\ &\Phi^{-1}((d_{ij})) = \sum_{i,j} e_{i1} d_{ij} e_{1j}. \end{aligned} \tag{5}$$

Both composites are the identity; multiplication is checked by inserting $\sum_\ell e_{\ell\ell} = 1$ between two factors.

Set $a_{ij} = e_{1i}te_{j1}$ and $b_{ij} = e_{1i}t^{-1}e_{j1}$. By (5), the matrices (a_{ij}) and (b_{ij}) are inverse. Thus (1) gives a homomorphism $g : R \to D$ taking x_{ij} to a_{ij} and y_{ij} to b_{ij}. Conversely the coproduct property defines

$$f : C \longrightarrow M_3(R), \qquad f(e_{ij}) = E_{ij}, \quad f(t) = X, \quad f(t^{-1}) = Y.$$

Restrict f to D and identify $E_{11}M_3(R)E_{11}$ with R; denote this map by f_0. On every generator, $f_0 g$ is the identity.

The map g is also onto. Every element of D is a sum of terms $e_{11}z_1 \cdots z_r e_{11}$, with each z_j a matrix unit, t, or t^{-1}. Inserting $\sum_\ell e_{\ell\ell} = 1$ between adjacent factors expresses each term as a sum of products of entries $e_{1i}z_j e_{\ell 1}$. An entry is 0 or e_{11} when z_j is a matrix unit; otherwise it is one of the $a_{i\ell}, b_{i\ell}$. These entries generate D, proving surjectivity. Since $f_0 g = 1_R$, the map g is an isomorphism.

Combining this with (5) identifies f with an isomorphism $C \cong M_3(R)$; it sends h to H, t to X, and ht to σ. Consequently (4) describes all of $\mathrm{GL}_3(R)$, not merely a subgroup of elementary matrices.

5. The free-product argument

For completeness, the required facts about free products are proved using their elementary coset tree. Let $L = P * Q$. The vertices are the disjoint union $L/P \sqcup L/Q$; an edge labelled $\lambda \in L$ joins λP to λQ. Reduced-word normal form shows that this graph is connected and has no cycles. Left multiplication acts without edge inversions, since it preserves the two vertex types. Edge stabilizers are trivial, since edge labels carry the left regular action.

It follows that a nonidentity element cannot fix two vertices: it would fix every edge of the path between them. A nonidentity involution a fixes exactly one vertex. Indeed, it preserves the finite path from any vertex v to av and reverses its endpoints, so fixes its midpoint. That midpoint cannot be the interior of an edge, since the action has no inversions. Uniqueness follows from trivial edge stabilizers.

Lemma 2. *If nonidentity involutions $a, b \in L$ have different fixed vertices, then ab translates a bi-infinite geodesic. In particular it has infinite order and fixes no vertex.*

Proof. Let v, w be the fixed vertices of a, b, respectively, and let $J = [v, w]$. Since a fixes no edge at v, the paths J, aJ meet only at v; similarly J, bJ meet only at w. Thus the path K from w through v to aw has length $2d(v, w) > 0$. Its translates $(ab)^m K$ join consecutively, since $(ab)w = aw$. At a joining point there is no backtracking: applying a^{-1} to the join between K and $(ab)K$, the incoming and outgoing directions become those from w toward v and toward bv. These directions differ because b fixes no edge at w. The same holds at every translate of this join. Hence these paths form a bi-infinite geodesic, translated by ab through distance $2d(v, w)$. Such a translation has infinite order. It fixes no vertex anywhere in the tree: a fixed vertex would have a fixed nearest point on this invariant geodesic, which is impossible. □

Lemma 3. *For $1 \neq p \in P$ and $1 \neq q \in Q$, the element pq fixes no vertex of the coset tree.*

Proof. Let e be the edge labelled 1. The sequence of edges

$$\ldots, (pq)^{-1}e, (pq)^{-1}pe, e, pe, (pq)e, (pq)pe, (pq)^2e, \ldots$$

forms a line. Consecutive edges meet alternately at cosets of P and Q, and there is no backtracking because p, q are nonidentity. Multiplication by pq shifts this line by two edges. As in the preceding proof, it therefore fixes no vertex. □

6. All transvections are covered

Proof of Theorem 1. Use $L = \mathrm{U}(C) = P * \langle t \rangle$, identified with $\mathrm{GL}_3(R)$ by f. The element $s = ht$, corresponding to σ, fixes no vertex by Lemma 3.

Transport arbitrary T_1, T_2 as in the theorem to $a_1, a_2 \in L$. In characteristic two,

$$T_i^2 = I + 2(T_i - I) + (T_i - I)^2 = I.$$

Thus both a_i are nonidentity involutions. Let v be the unique fixed vertex of a_1. The involution $s^{-1}a_1s$ has fixed vertex $s^{-1}v \neq v$. Lemma 2 implies that

$$c = [s, a_1] = (s^{-1}a_1s)a_1$$

is a translation, and hence fixes no vertex. If w is the unique fixed vertex of a_2, the fixed vertex of $c^{-1}a_2c$ is $c^{-1}w \neq w$. Applying the lemma again shows that

$$[c, a_2] = (c^{-1}a_2c)a_2$$

has infinite order. This proves the first assertion for every pair T_1, T_2, including matrices outside the elementary subgroup.

Finally, in any characteristic-two ring, a unipotent matrix $U = I + N$ has finite order: if $N^r = 0$, choose m with $2^m \geq r$. Repeated squaring gives

$$U^{2^m} = I + N^{2^m} = I.$$

The infinite-order double commutator is therefore not unipotent. Equivalently, $([[\sigma, T_1], T_2] - I)^r \neq 0$ for every positive integer r. □

Provenance and verification scope

This is an independent-verification draft, prepared from the original campaign certificate, with the mathematical argument made explicit. It is not human acceptance or a claim of completion of the repository's full adversarial-verification protocol. The package preserves the original certificate and the historical automated validator record separately from a verdict-free proof input and a scoped export review.

The load-bearing published result is Gerasimov's theorem [1]. His introduction also discusses the universal matrix-coefficient ring used in the construction; that construction is not presented here as new. The package includes a primary-source audit of the theorem and its hypotheses. No claim of literature novelty for the application to problem 10.46 is made. The primary journal record is available at MathNet (sm2646); the article DOI is `10.1070/SM1989v062n01ABEH003225`.

The symbolic argument above proves the universal quantifier over transvections. Supplementary Python checks verify two exact inverse identities in noncommutative polynomials over $\mathbb{F}_2$ and test 65,025 double commutators in a bounded model free product. That model is *not* the entire unit group in the proof and the scan is not an exhaustive transvection certificate. No step of the mathematical proof depends on the scan. The replay report records its environment and comparison with the preserved source output; there is no claim of an independent GAP replay.

References

[1] V. N. Gerasimov. The group of units of a free product of rings. *Mathematics of the USSR-Sbornik*, 62(1):41–63, 1989.