# AN IMPORTED COUNTEREXAMPLE TO KOUROVKA NOTEBOOK 11.23

Abstract. The imported argument proposes the order-four cyclic shift on $\Gamma^4$, where $\Gamma$ is the first Grigorchuk group, as an e-automorphism that is not a nil-automorphism. Its e-condition is established in the source by a group-theoretic argument using three external inputs; non-nilness has an exact six-stage section certificate. This draft formats the original proof for review. Historical validation and inherited bibliographic metadata are not a new verification or human acceptance.

## 1. Problem and conventions

Kourovka Notebook Problem 11.23 asks:

> An automorphism $\phi$ of a group $G$ is called a nil-automorphism if, for every $a \in G$, there exists $n$ such that $[a, {}_n\phi] = 1$. Here $[x, {}_1y] = [x, y]$ and $[x, {}_{i+1}y] = [[x, {}_iy], y]$. An automorphism $\phi$ is called an e-automorphism if, for any two $\phi$-invariant subgroups $A$ and $B$ such that $A \nsubseteq B$, there exists $a \in A \setminus B$ such that $[a, \phi] \in B$. Is every e-automorphism of a group a nil-automorphism?

The statement is attributed to V. G. Vilyatser. Line wraps and iterated commutator subscripts are normalized from the retained Notebook extraction.

The witness is $G = \Gamma^4$, where $\Gamma$ is the first Grigorchuk group, and

$$\phi(g_0, g_1, g_2, g_3) = (g_1, g_2, g_3, g_0).$$

The binary rooted tree has vertices the finite words in $\{0, 1\}$. Products of tree automorphisms act on the left: $(gh)(w) = g(h(w))$. Define

$$a = (1, 1)(0\ 1), \qquad b = (a, c), \qquad c = (a, d), \qquad d = (1, b),$$

and $\Gamma = \langle a, b, c, d\rangle$ inside the automorphism group of this tree. These recursions specify the action on every finite word and hence specify the group without reference to a database. For an automorphism $\alpha$ put $\delta_\alpha(x) = x^{-1}\alpha(x)$. All iterated commutators below use precisely this convention. The automorphism $\phi$ has order four.

## 2. The group-theoretic input

The source uses the following established results about this particular $\Gamma$. The references and their claimed scope are inherited from that source.

**(T1)** $\Gamma$ is an infinite torsion 2-group and is just infinite. The normal subgroup $K = \langle [a, b]\rangle^\Gamma$ has index 16. In the first-level section embedding, $K$ contains the supported copies $K \times K$ with finite index. These facts imply that, for any finite complete binary prefix code $P$, the product of supported copies of $K$ on the cylinders $v\{0, 1\}^{\mathbb{N}}$, $v \in P$, is a finite-index subgroup of $\Gamma$. In particular $K^4$ embeds in $\Gamma$, with finite-index image, by putting its four factors below the four vertices of level two.

**(T2)** Grigorchuk–Wilson: every infinite finitely generated subgroup of $\Gamma$ is abstractly commensurable with $\Gamma$; that is, the two groups have isomorphic subgroups of finite index.

**(T3)** Rover: the natural abstract commensurator of $\Gamma$ is its group $V_\Gamma$ of piecewise-$\Gamma$ homeomorphisms of the binary boundary. Explicitly, its elements have finite complete prefix codes $P, Q$, a bijection $\pi : P \to Q$, and labels $g_v \in \Gamma$, acting by $vw \mapsto \pi(v)g_v(w)$. The commensurator identification is by conjugation on finite-index subgroups. In particular, a commensuration of order dividing four is represented by such a homeomorphism of order dividing four.

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For (T1), the source cites [1, Section 1.6.1, Proposition 1.25, and Chapter 5, Theorem 5.2]. The displayed generator recursions there are the recursions above, with binary symbols relabelled $1, 2$ instead of $0, 1$. Torsion is recorded in Section 1.6.1. Just infiniteness also follows from Theorem 5.2: the branch factors are finite-index copies of $K$, which are finitely generated torsion groups, and thus have finite abelianizations. For (T2), the source cites [2]; for (T3), it cites [3]. The piecewise definition and the identification for exactly the displayed $\Gamma$ are also recorded in [4, Definition 3.2 and Example 3.3]. All further arguments are given below. In particular, no assertion about arbitrary infinite invariant subgroups is assumed from prior work.

## 3. Finite-index generation and the e-condition

**A useful contraction observation.** The relations $a^2 = b^2 = c^2 = d^2 = bcd = 1$ follow from the defining tree actions. After using them, any word alternates between $a$ and a letter in $\{b, c, d\}$. If its reduced length is $n \geq 2$, each first-level section has a word of length at most $\lceil n/2 \rceil$, since only letters $b, c, d$ contribute section letters and each contributes at most one per section. Iterating shows that the sections at all sufficiently deep vertices lie in $\{1, a, b, c, d\}$. This holds for any specified finite collection of elements, with a common depth. Every member of that set has order at most two.

**Lemma 1** (Finite-order piecewise actions can be refined)**.** *Let $z \in V_\Gamma$ satisfy $z^4 = 1$. There exists a finite complete prefix code $P$ whose cylinders are permuted by $z$, on which all local labels belong to $\Gamma$, and for which the return label around each cylinder cycle has order at most two.*

*Proof.* First choose a finite partition into cylinders so fine that every $z^i$, $0 \leq i < 4$, maps each cylinder by a prefix replacement followed by a $\Gamma$ tree automorphism. The four image partitions consist of cylinders. Their common refinement is invariant under $z$: it is the partition into atoms of a finite family permuted by $z$. Nonempty intersections of cylinders are cylinders. The local labels remain in $\Gamma$ by closure under sections.

On a cycle of $\ell$ cylinders, $\ell$ divides 4, and the return map is a tree automorphism $g \in \Gamma$ with $g^{4/\ell} = 1$. Subdivide every cylinder to the same additional depth $n$, chosen so that all depth-$n$ sections of each of the finitely many powers $g^j$, $0 \leq j \leq 4/\ell$, lie in $\{1, a, b, c, d\}$. Include the return maps based at every old cylinder when making this choice, and do this for all cycles at once. A new cylinder in that cycle has period $\ell q$, where $q$ is its suffix's orbit length under $g$. Its return label is the section $(g^q)|_u$, so has order at most two. The refinement is again permuted by $z$. □

**Lemma 2** (Finite-index generation by elements with terminating commutators)**.** *Let $H$ be an infinite group abstractly commensurable with $\Gamma$, and let $\alpha$ be an automorphism of $H$ with $\alpha^4 = 1$. Then $H$ has a finite-index subgroup generated by elements $x$ such that $\delta_\alpha^n(x) = 1$ for some $n$.*

*Proof.* Choose $H_0 \leq H$ of finite index isomorphic to a finite-index subgroup of $\Gamma$. Replace it by the intersection of its four $\alpha$-translates. The resulting $N$ is $\alpha$-invariant and has finite index in $H$, and an isomorphism $f : N \to L$ identifies it with a finite-index subgroup $L$ of $\Gamma$. The automorphism $\beta = f \alpha f^{-1}$ of $L$ defines a commensuration with fourth power one. By (T3) it is represented by a homeomorphism $z \in V_\Gamma$ with $z^4 = 1$. After restricting to a finite-index subgroup of $L$, $\beta$ agrees with conjugation by $z$. Intersect that subgroup with its four $\beta$-translates. We may therefore replace $N, L$ by finite-index subgroups and assume exactly that $zLz^{-1} = L$ and $\beta(h) = zhz^{-1}$ on $L$. This restriction changes neither finite index in $H$ nor the automorphism under consideration.

Choose the $z$-invariant prefix code $P$ from Lemma 1. For a vertex $v$ and $h \in K$, write $e_v(h)$ for the tree automorphism acting as $h$ below $v$ and as the identity elsewhere. By (T1), $e_v(K) \leq \Gamma$. For each $v \in P$ the subgroup

$$E_v = \{h \in K : e_v(h) \text{ belongs to } L\}$$

has finite index in $K$, since $L$ has finite index in $\Gamma$. Thus the intersection of the finitely many $E_v$ has finite index in $\Gamma$. Take its core in $\Gamma$, denoted $C$. Then $C$ is normal of finite index in $\Gamma$, $C \leq K$, and the product of all $e_v(C)$, $v \in P$, lies in $L$.

Let $J$ be the subgroup of $C$ generated by all its involutions. The group $C$ is infinite and torsion 2, so it has a nonidentity involution. Moreover $J$ is characteristic in $C$ and hence normal in $\Gamma$. Just infiniteness in (T1) therefore implies that $J$ has finite index in $\Gamma$. In particular $J \leq C$ is generated by involutions and is normal in $\Gamma$. Set

$$D = \prod_{v \in P} e_v(J).$$

The product is direct because the cylinders are disjoint. It lies in $L$ and has finite index in $\Gamma$, and hence in $L$: the corresponding product of copies of $K$ has finite index by (T1), and $J$ has finite index in $K$. All local labels of $z$ lie in $\Gamma$ and therefore preserve $J$ by conjugation. It follows that $z$ normalizes $D$.

We show that every coordinate involution $e_v(h)$, $h \in J$, $h^2 = 1$, has terminating commutators with conjugation by $z$. Work on the finite cycle of cylinders containing $v$. By transporting the coordinate identifications along the local $\Gamma$ labels, conjugation by $z$ becomes a cyclic permutation of $\ell$ copies of $J$, with the last transition twisted by $\tau =$ conjugation by the return label $g$. Lemma 1 gives $g^2 = 1$, so $\tau^2 = 1$. Under repeated application of $z$, a coordinate involution has, in each coordinate, at most the two transported values $h$ and $\tau(h)$. Both are involutions. The group $\langle h, \tau(h) \rangle$ is finite dihedral (possibly of order two): the product $h\tau(h)$ has finite 2-power order because it belongs to $\Gamma$. Consequently the subgroup $F$ generated by the entire $z$-orbit of $e_v(h)$ embeds in a finite product of finite dihedral 2-groups. It is a finite 2-group, is $z$-invariant, and $z$ acts on it with order dividing four. The semidirect product $F \rtimes C_4$ is a finite 2-group and hence is nilpotent. Its lower central series shows that $\delta_\beta^n(e_v(h)) = 1$ for some $n$. All the hypotheses of this elementary finite-$p$-group fact have thus been checked: $F$ is finite of 2-power order and the action has 2-power order.

These coordinate involutions generate $D$. Transporting $D$ back by $f$ gives the finite-index subgroup of $H$ required in the lemma. $\square$

**Lemma 3** (The finite-index generation criterion gives the e-condition)**.** *Let $H$ be a torsion 2-group, $\alpha^4 = 1$, and suppose $H$ has a finite-index subgroup generated by $\alpha$-Engel elements (elements with terminating $\delta_\alpha$ iteration). For every proper $\alpha$-invariant subgroup $B < H$ there is $x \in H \setminus B$ with $\delta_\alpha(x) \in B$.*

*Proof.* Suppose otherwise. Every $\alpha$-Engel element must be in $B$: if one were outside, its iterates, ending at $1 \in B$, would have a last term outside $B$ whose next term lies in $B$, contrary to the supposition. Thus $B$ contains the stated finite-index subgroup, and $[H : B]$ is finite.

The index $[H : B]$ is a power of two. Indeed, the image of $H$ in its finite permutation action on the left cosets $H/B$ is a finite group all of whose elements have 2-power order; Cauchy's theorem makes it a finite 2-group. The index, an orbit size, is a power of two and is even since $B$ is proper. The automorphism $\alpha$ permutes $H/B$. Its cycles have lengths $1, 2$ or $4$. There are an even number of fixed cosets, and the coset $B$ is fixed, so some $xB$ other than $B$ is fixed. The equation $\alpha(x)B = xB$ says exactly $x^{-1}\alpha(x) \in B$. This contradiction proves the lemma. No normality of $B$ was used. $\square$

**Verification of the entire e-condition for the witness.** Let $A$ and $B$ be arbitrary $\phi$-invariant subgroups of $G$ with $A \not\subseteq B$. Choose $u \in A \setminus B$ and put

$$H = \langle u, \phi(u), \phi^2(u), \phi^3(u) \rangle, \qquad B_0 = H \cap B.$$

Then $H$ is finitely generated, $\phi$-invariant, contained in $A$, and $B_0$ is a proper $\phi$-invariant subgroup of $H$. It suffices to find $x \in H \setminus B_0$ with $\delta_\phi(x) \in B_0$.

If $H$ is finite, $H \rtimes C_4$ is a finite 2-group, so every element of $H$ has terminating $\delta$ iteration, and the required $x$ is obtained by taking the last iterate outside $B_0$.

If $H$ is infinite, $H \cap K^4$ has finite index in $H$ and is finitely generated (Schreier's theorem). The level-two embedding $K^4 \to \Gamma$ from (T1) maps it injectively to an infinite finitely generated subgroup of $\Gamma$. (T2) applies to exactly this subgroup, so $H$ is abstractly commensurable with $\Gamma$. The restriction of $\phi$ to $H$ has fourth power one. Lemma 2 supplies a finite-index subgroup generated by $\phi$-Engel elements, and Lemma 3 applies because $H \leq \Gamma^4$ is a torsion 2-group. It gives the required $x$. Thus the e-condition holds for every pair $A, B$, including infinite nonnormal and infinite-index subgroups.

## 4. Verification that $\phi$ is not a nil-automorphism

Put $u = (d, ba, acab, ba)$. For a four-tuple $w$ define

$$\Delta(w) = (w_0^{-1}w_1, w_1^{-1}w_2, w_2^{-1}w_3, w_3^{-1}w_0).$$

This is exactly $\delta_\phi$. The following is a finite exact certificate. Every row says that $\Delta^k$(source) fixes the indicated first-level vertex in all four coordinates and that its coordinatewise section there is the next row's source, with the last row returning to the first:

| Source | $k$ | Vertex |
|---|---|---|
| $(d, ba, acab, ba)$ | 2 | 0 |
| $(dada, adacac, caca, ad)$ | 2 | 0 |
| $(ababad, dabadad, da, ca)$ | 1 | 0 |
| $(cabab, ba, c, dac)$ | 2 | 0 |
| $(bac, ca, abaca, acabab)$ | 1 | 1 |
| $(d, aca, abac, cadad)$ | 1 | 0 |

Consequently $\Delta^9(u)$ fixes $v = 000010$ coordinatewise and has section $u$ at $v$. All these identities are verified using only the displayed tree recursions and $a^2 = b^2 = c^2 = d^2 = bcd = 1$. The script `verify_section_cycle.py` replays each row and the combined nine-step identity. The separate standalone verifier `verify_witness_standalone.py` checks the combined identity without the search code or its word-problem routines.

Sections on a vertex stabilizer are homomorphisms and commute with the coordinate shift. Induction now gives that $\Delta^{9r}(u)$ fixes $v$ repeated $r$ times and has section $u$ there, for every $r \geq 1$. The tuple $u$ is nonidentity: its first coordinate $d$ interchanges 100 and 101. Thus $\Delta^{9r}(u)$ is nonidentity for every $r$, so no $\Delta^n(u)$ can be identity. This falsifies the nil-automorphism conclusion.

**Reproduction and scope.** The original proof specifies Python 3 (checked there on Python 3.12.14), the standard library only, and no floating-point arithmetic. Use the repository's `uv run` commands and fresh-directory instructions in the accompanying experiment README; the second checker writes a result file relative to its working directory. The e-condition is established by the preceding mathematical argument, not by finite-subgroup enumeration. The only substantial cited inputs are (T1)–(T3); the source's claim is that their specific hypotheses and all transfers to $H, L, C, J, D$ have been checked above. The initial searches that failed to find section cycles are not used as evidence for this counterexample.

## Methods and provenance

This is a faithful typesetting of `counterexample.txt` from solver 1 of run `counterexample-only-astra-all-20260906-01-11.23-973055f727-e1-r2`, configured with `gpt-6-astra` and `ultra` reasoning. The exact original proof, eight submitted mathematical artifacts in total, task, solver/validator records, and validator stage program are retained with source-relative paths. Their 12 source/copy SHA-256 pairs are in `experiments/imported-counterexample/manifest.json`.

The original runner recorded `SOLVED_VALIDATED`; its validator recorded `PASS`, reporting the standalone and its own six-stage replay. That validator explicitly did not execute the solver's `verify_section_cycle.py`. On September 14, 2026, the coordinating task freshly replayed all three programs successfully: all six stages, the combined nine-iteration identity, and seven historical manifest hashes passed. Fresh outputs are under `experiments/replay-20260914/`. Certificate status is COMPLETE (6/6 stages and the combined identity).

These computations establish the stated finite section certificate only. They do not verify the universal e-condition, the external theorem inputs, novelty, or GAP corroboration. Bibliographic entries below reproduce only metadata in the original proof and are explicitly inherited, not freshly audited. No new theorem, proof repair, or citation is introduced by this formatting. Historical validation is neither this repository's candidate qualification nor human acceptance; its full verification gate remains pending. The original proof separates the known inputs (T1)–(T3) from its ensuing lemmas and argument; this import makes no independent novelty claim for that argument.

## References


[1] L. Bartholdi, R. I. Grigorchuk, Z. Sunik, *Branch Groups*, Section 1.6.1, Proposition 1.25, and Chapter 5 (Theorem 5.2). `https://people.tamu.edu/~grigorch/publications/handbook.pdf`.

[2] R. I. Grigorchuk and J. S. Wilson, *A Structural Property Concerning Abstract Commensurability of Subgroups*, J. London Math. Soc. **68** (2003), 671–682. DOI: 10.1112/S0024610703004745.

[3] C. E. Rover, *Abstract Commensurators of Groups Acting on Rooted Trees*, Geometriae Dedicata **94** (2002), 45–61. DOI: 10.1023/A:1020916928393.

[4] V. Nekrashevych, *Finitely presented groups associated with expanding maps*, Definition 3.2 and Example 3.3. `https://arxiv.org/pdf/1312.5654`.