AN EQUATION-COUNTING CRITERION FOR SIMULTANEOUS DEFECT ZERO IN KOUROVKA PROBLEM 11.99

Abstract. For a finite group G we express the number of irreducible complex characters having defect zero at more than one prime as a rational linear combination of counts of solutions of explicit commutator equations. All coefficients depend only on $|G|$, and $\lfloor\sqrt{|G|}\rfloor$ equation counts suffice. Positivity of this expression gives a necessary and sufficient condition. The formula is a specialization of classical commutator-moment reconstruction to a simultaneous-prime selector.

1. The problem and the formula

Problem 11.99 of S. P. Strunkov asks [2, p. 54]:

> Find (in group-theoretic terms) necessary and sufficient conditions for a finite group to have complex irreducible characters having defect 0 for more than one prime number dividing the order of the group. Express the number of such characters in the same terms.

We count individual characters which have defect zero simultaneously at at least two primes. For $N = |G|$ and $\chi \in \mathrm{Irr}(G)$, defect zero at $p \mid N$ means $v_p(\chi(1)) = v_p(N)$, where v_p is the exponent of p in a positive integer. The criterion below uses only N, rational arithmetic and solution counts for group equations. It makes no efficiency or structural classification claim.

Set $D = \lfloor\sqrt{N}\rfloor$ and $[x, y] = xyx^{-1}y^{-1}$. Define

$$C_g(G) = \#\{(x_1, y_1, \ldots, x_g, y_g) \in G^{2g} : [x_1, y_1]\cdots[x_g, y_g] = 1\}, \qquad 1 \le g \le D. \tag{1}$$

The following data depend on N alone:

$$T_N = \{d \in \{1, \ldots, D\} : \#\{p \mid N : p \text{ prime},\ v_p(d) = v_p(N)\} \ge 2\}, \tag{2}$$

$$L_d(X) = \prod_{\substack{1 \le e \le D \\ e \ne d}} \frac{X - e^{-2}}{d^{-2} - e^{-2}}, \qquad 1 \le d \le D, \tag{3}$$

$$H_N(X) = \sum_{d \in T_N} L_d(X) = \sum_{j=0}^{D-1} h_j X^j. \tag{4}$$

Empty sums and products have values 0 and 1, respectively. The nodes d^{-2} are distinct, so all denominators are nonzero.

Theorem 1. *For every finite group G, the number of irreducible complex characters having defect zero for more than one prime divisor of $|G|$ is*

$$A(G) = \sum_{j=0}^{D-1} h_j \frac{C_{j+1}(G)}{N^{2j+1}}. \tag{5}$$

Consequently, the necessary and sufficient condition is $A(G) > 0$. Each qualifying character is counted once, even if it has defect zero at three or more primes. If $N = 1$ or N is a prime power, $A(G) = 0$.

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The classical input is Frobenius's commutator-count formula, in the form given in [1, Theorem 1.1]. Reconstruction of the multiset of irreducible degrees from commutator counts is already due to Robinson [3]. The present argument specializes this method to the predicate in (2) and writes the resulting finite count explicitly. No novelty is claimed for reconstruction itself.

2. Proof

For a finite group and every $g \geq 1$, Frobenius's identity is

$$C_g(G) = N^{2g-1} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{2-2g}. \tag{6}$$

We recall its algebraic derivation to fix the normalization and the commutator convention. In $\mathbb{C}[G]$ let

$$z = \sum_{x,y \in G} [x, y].$$

For an irreducible representation ρ of degree d and character χ, Schur's lemma and the trace give, for every $d \times d$ matrix B,

$$\sum_{x \in G} \rho(x) B \rho(x)^{-1} = \frac{N}{d} \operatorname{tr}(B) I_d.$$

Indeed, the left side commutes with $\rho(G)$ and its trace is $N \operatorname{tr}(B)$. Applying this with $B = \rho(y)$ yields

$$\rho(z) = \frac{N}{d} \sum_{y \in G} \chi(y) \rho(y)^{-1}.$$

The final sum is scalar because χ is constant on conjugacy classes. Its trace is $\sum_y \chi(y)\chi(y^{-1}) = N$, by complex character orthogonality. Hence $\rho(z) = (N^2/d^2) I_d$. The regular representation contains each ρ with multiplicity d, so the trace of left multiplication by z^g is

$$\sum_{\chi \in \mathrm{Irr}(G)} d^2 \left(\frac{N^2}{d^2} \right)^g, \qquad d = \chi(1).$$

In the group-element basis, every diagonal coefficient is the identity coefficient of z^g, namely $C_g(G)$. Equating this trace with $N C_g(G)$ proves (6). The regular dimension identity also gives

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = N, \qquad 1 \leq \chi(1) \leq D. \tag{7}$$

These are the same two identities recorded in [1, Theorem 1.1 and the following paragraph].

Proof of Theorem 1. Lagrange interpolation gives

$$L_d(e^{-2}) = \begin{cases} 1, & e = d, \\ 0, & e \neq d, \end{cases} \qquad 1 \leq d, e \leq D.$$

By (7), $H_N(\chi(1)^{-2})$ is therefore 1 exactly when the character χ satisfies the simultaneous-prime condition, and is 0 otherwise. Substituting (6) with $g = j+1$ in (5) and interchanging finite sums gives

$$\begin{aligned} A(G) &= \sum_{\chi \in \mathrm{Irr}(G)} \sum_{j=0}^{D-1} h_j \chi(1)^{-2j} \\ &= \sum_{\chi \in \mathrm{Irr}(G)} H_N(\chi(1)^{-2}). \end{aligned}$$

This is precisely the required count. The selector acts on each character before summation, so no prime-pair multiplicity occurs. If $N = 1$ or N is a prime power, T_N is empty. □

All terms on the right of (5) are computed from the multiplication law of G; a character table or degree list is not an input. If an integer inequality is desired, multiply $A(G) > 0$ by a positive common denominator of the rational coefficients h_j/N^{2j+1}.

Open questions

A structural characterization of the groups in the question, or a criterion using substantially fewer equation counts, would strengthen this equation-counting answer.

Methods and AI disclosure

This internal candidate was developed and typeset in Codex using `gpt-6-astra`. Its argument, including the group-algebra derivation, passed two independent adversarial reviews before typesetting. The citation audit checked the Frobenius formula and its conventions against the author-hosted source, and authenticated the historical reconstruction attribution. Independently implemented Python and GAP calculations agreed on nine groups, including a case qualifying at three primes; these computations are corroboration, not proof premises. Searches found no exact prior treatment of this application, without establishing priority or editorial acceptance. Review reports, source notes, code and transcripts are retained in the accompanying Kourovka repository under `problems/11.99/`. Human mathematical acceptance remains pending.

References

1. Avraham Aizenbud, Nir Avni, and Yoav Krauz, *Relative Frobenius formula*, 2016, arXiv:1603.06190. Author-hosted manuscript dated December 21, 2015, `https://www.wisdom.weizmann.ac.il/~aizenr/4Publications/Rel_Frob.pdf`.
2. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory. The Kourovka Notebook*, 2026, arXiv:1401.0300v45, `https://arxiv.org/abs/1401.0300v45`.
3. Geoffrey R. Robinson, *Characters and the commutator map*, J. Algebra **321** (2009), no. 11, 3521–3526.