A FINITE COUNTEREXAMPLE TO KOUROVKA NOTEBOOK 13.19

Abstract. An imported construction gives subgroups $H \trianglelefteq Q \leq \mathrm{UT}_3(\mathbb{F}_2)^4$ such that both project onto every factor and Q/H is a nonregular 2-group of order 32. The argument and its original exact checker are preserved for review. This typeset draft does not constitute completion of the repository's verification protocol.

1. Problem and construction

Kourovka Notebook Problem 13.19, posed by Yu. M. Gorchakov, asks:

> Suppose that both a group Q and its normal subgroup H are subgroups of the direct product $G_1 \times \cdots \times G_n$ such that for each i the projections of both Q and H onto G_i coincide with G_i. If Q/H is a p-group, is Q/H a regular p-group?

All arithmetic below is in $\mathbb{F}_2$. Take $p = 2$ and $n = 4$, and let every factor be $G = \mathrm{UT}_3(\mathbb{F}_2)$, whose elements are

$$g(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{F}_2.$$

This is a subgroup of $\mathrm{GL}_3(\mathbb{F}_2)$ of order 8, with

$$g(\alpha, \beta, \gamma)g(\delta, \varepsilon, \varphi) = g(\alpha + \delta, \beta + \varepsilon, \gamma + \varphi + \alpha\varepsilon),$$
$$g(\alpha, \beta, \gamma)^{-1} = g(\alpha, \beta, \gamma + \alpha\beta), \qquad 1 = g(0, 0, 0).$$

For $q = (g(\alpha_i, \beta_i, \gamma_i))_{i=1}^4 \in G^4$, define

$$Q = \left\{ q : \sum_i \alpha_i = \sum_i \beta_i = 0 \right\},$$
$$H = \left\{ q : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4, \quad \beta_1 = \beta_2 = \beta_3 = \beta_4, \quad \sum_i \gamma_i = 0 \right\}.$$

2. Verification of the hypotheses

The map $G^4 \to \mathbb{F}_2^2$ given by $q \mapsto (\sum_i \alpha_i, \sum_i \beta_i)$ is a surjective homomorphism by the multiplication formula. Its kernel is Q, so $|Q| = 8^4/4 = 1024$.

An element of H has constant first coordinates A and constant second coordinates B. It lies in Q because $4A = 4B = 0$. For elements of H with constants (A, B) and (D, E), the product has constants $(A + D, B + E)$ and central-coordinate sum $0 + 0 + 4AE = 0$. Inversion adds AB to every central coordinate, changing their sum by $4AB = 0$. Thus H is a subgroup of Q. There are two choices for each of A, B and 2^3 choices for the central coordinates, giving $|H| = 2^5 = 32$.

For $q = (g(\alpha_i, \beta_i, \gamma_i)) \in Q$ and $h = (g(A, B, d_i)) \in H$, direct matrix multiplication gives

$$q^{-1}hq = (g(A, B, d_i + A\beta_i + \alpha_i B))_{i=1}^4.$$

The central coordinates now sum to

$$\sum_i d_i + A \sum_i \beta_i + B \sum_i \alpha_i = 0,$$

so $H \trianglelefteq Q$.

For every $s \in G$, the diagonal tuple (s, s, s, s) lies in H: its first two coordinates are constant and its central-coordinate sum is four times one element of $\mathbb{F}_2$, hence zero. Therefore H projects

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onto each factor G_i, and so does Q, since $H \le Q$. The quotient $P = Q/H$ has order $1024/32 = 32 = 2^5$ and is a finite 2-group. The stated problem does not exclude $p = 2$.

3. Explicit failure of regularity

Put $a = g(1,0,0)$, $b = g(0,1,0)$, and $t = g(0,0,1)$. The multiplication formula gives $a^2 = b^2 = t^2 = 1$ and $(ab)^2 = t$. Define

$$x = (a, a, 1, 1), \qquad y = (b, 1, b, 1), \qquad z = (t, 1, 1, 1).$$

All three belong to Q, while $z \notin H$ because its central-coordinate sum is 1. With the convention $[u, v] = u^{-1}v^{-1}uv$, direct multiplication gives

$$x^2 = y^2 = 1, \qquad (xy)^2 = z, \qquad [x, y] = z.$$

For $X = xH$, $Y = yH$, $Z = zH$ in P, it follows that

$$X^2 = Y^2 = 1, \qquad (XY)^2 = Z \neq 1, \qquad Z^2 = 1.$$

For two arbitrary factor elements, the commutator is

$$[g(\alpha, \beta, \gamma), g(\delta, \varepsilon, \varphi)] = g(0, 0, \alpha\varepsilon + \delta\beta).$$

Thus G' is contained in the central subgroup $\{1, t\}$, of exponent 2. Consequently $(G^4)'$ is central of exponent 2. The same exponent bound holds for Q' and its image P', so every element of $\langle X, Y\rangle'$ has square 1.

A finite p-group is regular only if, for each pair U, V, the element $(UV)^p$ can be written as U^pV^p times a product of pth powers of elements of $\langle U, V\rangle'$. For X, Y and $p = 2$, every such right-hand side is 1, whereas $(XY)^2 = Z \neq 1$. Hence P is not regular. This supplies the counterexample directly through the defining regularity identity, without a classification theorem.

Computation, sources, and provenance

The unchanged original `witness.txt` and `verify_witness.py` are under `experiments/imported-counterexample/source/`, relative to the problem directory. The script uses only the Python standard library and encodes $g(\alpha, \beta, \gamma)$ as $\alpha + 2\beta + 4\gamma$. Its archived certificate reports `COMPLETE_COUNTEREXAMPLE` under Python 3.12.14: all 4096 ambient elements, $1024^2 = 1\,048\,576$ products in Q and their quotient images, $1024 \cdot 32 = 32\,768$ conjugations, every coordinate projection, and $32^3 = 32\,768$ quotient associativity checks. It also checks the factor law against binary matrices, constructs all quotient cosets, and checks the derived subgroups and the displayed regularity failure. A fresh replay in `experiments/replay-20260914/` completed with exit code 0 and the same mathematical counts. It reproduces the original script; it is not an independent algorithm or a GAP check. Both historical and fresh executions are COMPLETE (1/1 program). Replay instructions accompany the archive.

Solver 1 of the archived campaign was configured with `gpt-6-astra` and `ultra` reasoning. Its validator recorded PASS and a byte-identical replay of the same script; this was not a second implementation. The runner recorded `SOLVED_VALIDATED` on September 12, 2026. The import manifest preserves the selected source hashes. These historical labels confer neither repository `candidate` status nor human acceptance. The repository's independent-verifier, citation-audit, and GAP requirements remain pending.

This draft formats the original proof without adding a result. Scalar matrix entries have been changed from Latin to Greek letters to distinguish them from the factor elements a, b; a generic factor element is denoted by s. The original cites Hobby [1] for the regularity definition and its applicability at $p = 2$, and states that no theorem from that paper is needed. The following bibliographic metadata is inherited from that original witness and has not been independently checked during this import.

References

[1] C. R. Hobby, *Nearly regular p-groups*, Canadian Journal of Mathematics **19** (1967), 520–522, doi:10.4153/CJM-1967-044-8.