ON BARDAKOV'S PROBLEM 14.15: THE COMMUTATOR WIDTH OF $\operatorname{Aut} F_n$ IS INFINITE

Abstract. Problem 14.15 of the Kourovka Notebook (V. G. Bardakov, 1999) asks for the supremum k_n of the commutator lengths of the elements of the derived subgroup of $\operatorname{Aut} F_n$, $n \geq 3$. We show that $k_n = \infty$ for every $n \geq 3$: there is a single commutator $c \in \operatorname{Aut} F_n$ with $\operatorname{cl}(c^m) \to \infty$. The proof pulls a homogeneous quasimorphism of positive defect back from $\operatorname{Out} F_n$ (Bestvina–Feighn; alternatively Osin and Bestvina–Bromberg–Fujiwara) and applies the elementary half of Bavard duality. The same conclusion holds for commutators taken inside $[\operatorname{Aut} F_n, \operatorname{Aut} F_n] = \operatorname{SAut} F_n$ and for $\operatorname{Out} F_n$, and it also follows in two lines from the verbal-width theorem of Bestvina–Bromberg–Fujiwara for acylindrically hyperbolic groups. Nothing here is deep; the point is that the question, still listed as open, is answered by published theorems.

1. The problem

Problem 14.15 of the Kourovka Notebook [8], posed by V. G. Bardakov in the 14th issue (1999), reads:

> For the automorphism group $A_n = \operatorname{Aut} F_n$ of a free group F_n of rank $n \geq 3$ find the supremum k_n of the commutator lengths of the elements of A'_n.
>
> It is easy to show that $k_2 = \infty$. On the other hand, the commutator length of any element of the derived subgroup of $\varinjlim A_n$ is at most 2 (R. K. Dennis, L. N. Vaserstein [6]).

Notation. F_n is free of rank n, $A_n = \operatorname{Aut} F_n$, $A'_n = [A_n, A_n]$, $O_n = \operatorname{Out} F_n$, and $\pi \colon A_n \to O_n$ is the quotient map. For a group G and $g \in G' = [G,G]$, the *commutator length* $\operatorname{cl}_G(g)$ is the least k such that g is a product of k commutators $[x_i, y_i]$ with $x_i, y_i \in G$; it is finite on G' by definition. The quantity asked for is

$$k_n \;=\; \sup\{\operatorname{cl}_{A_n}(g) : g \in A'_n\} \;\in\; \mathbb{N} \cup \{\infty\},$$

the commutator width of A_n; "$k_n = \infty$" means that cl_{A_n} is unbounded on A'_n, never that some element has infinite commutator length. Commutator conventions are immaterial: $x^{-1}y^{-1}xy = [x^{-1}, y^{-1}]$ in the convention $[a,b] = aba^{-1}b^{-1}$ and conversely, so "product of k commutators" means the same thing in both. We write $\operatorname{SAut} F_n$ for the kernel of $A_n \to \mathrm{GL}_n(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\}$.

The statement admits one small ambiguity: the commutators may be taken among elements of A_n (as written) or among elements of A'_n itself. For $n \geq 3$, $A'_n = \operatorname{SAut} F_n$ is perfect (Lemma 2.5), so the second reading makes sense, and $\operatorname{cl}_{A'_n} \geq \operatorname{cl}_{A_n}$ on A'_n because a commutator of elements of A'_n is a commutator of elements of A_n. Hence $k_n = \infty$ under the first reading implies it under the second; Theorem 1.1 settles both.

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Main result.

Theorem 1.1. *Let $n \geq 3$.*

(a) *There are $a, b \in A_n$ such that $c = [a,b]$ satisfies $\mathrm{cl}_{A_n}(c^m) \geq \frac{1}{2}(\lambda m + 1)$ for all $m \geq 1$, for some constant $\lambda = \lambda(c) > 0$. In particular $k_n = \infty$.*

(b) *The same c satisfies $\mathrm{cl}_{A_n'}(c^m) \geq \frac{1}{2}(\lambda m + 1)$, and $\pi(c) \in O_n'$ satisfies $\mathrm{cl}_{O_n}(\pi(c)^m) \geq \frac{1}{2}(\lambda m + 1)$. So the commutator widths of $\mathrm{SAut}\, F_n$ and of $\mathrm{Out}\, F_n$ are infinite as well.*

In particular $\mathrm{scl}_{A_n}(c) \geq \lambda/2 > 0$: the stable commutator length of A_n does not vanish on A_n'. This is consistent with the notebook's remark about $\varinjlim A_n$: Dennis–Vaserstein's bound and Kotschick's vanishing of scl on $\bigcup_n \mathrm{Aut}\, F_n$ [9, Thm. 3.6] are Mazur-swindle arguments that use unboundedly many commuting conjugate copies of A_n inside the limit; no such copies exist in a fixed A_n, and the two statements say nothing about k_n.

What is classical and what is new. Every input is published. The one deep fact is that O_n carries a homogeneous quasimorphism that is not a homomorphism: this is Bestvina–Feighn's $\dim \widetilde{QH}(\mathrm{Out}\, F_n) = \infty$ [4, Cor. 4.30], obtained from the action of $\mathrm{Out}\, F_n$ on a hyperbolic complex, and it also follows from Osin's acylindrical hyperbolicity of $\mathrm{Out}\, F_n$ [11, §8(b)] together with Bestvina–Bromberg–Fujiwara [2, Cor. 1.2]. The passage from such a quasimorphism to unbounded commutator length is the easy half of Bavard duality [1, 5] and is written out in full; the identification $A_n' = \mathrm{SAut}\, F_n$ uses one relator of Gersten's presentation [7]. What is new is only the assembly, and the observation that it closes a question the notebook still lists as open. Indeed the answer is even a two-line corollary of a 2019 theorem: Bestvina–Bromberg–Fujiwara prove that every verbal subgroup $w(G)$ with $d(w) \neq 1$ has infinite width in an acylindrically hyperbolic group G [3, Thm. 1.1]; applied to $G = \mathrm{Out}\, F_n$ and the commutator word this says that $\mathrm{Out}\, F_n$ has infinite commutator width, and $k_n = \infty$ follows by monotonicity (Second proof in Section 2; it does not give the quantitative form of (a)). Neither paper mentions $\mathrm{Aut}\, F_n$; [3] names $\mathrm{Out}\, F_n$ once, in its abstract, as an example of the ambient class, so the instantiation to the commutator word of $\mathrm{Out}\, F_n$ is one line away but not written. We found no statement of the value of k_n in the literature; a specialist may nevertheless regard Theorem 1.1 as folklore, and we make no claim of depth.

2. Proof

2.1. **Quasimorphisms.** We follow Calegari [5, Ch. 2]. A *quasimorphism* on a group G is a function $\varphi\colon G \to \mathbb{R}$ with finite *defect* $D(\varphi) = \sup_{a,b} |\varphi(ab) - \varphi(a) - \varphi(b)|$; it is *homogeneous* if $\varphi(a^m) = m\varphi(a)$ for all $a \in G$, $m \in \mathbb{Z}$. $Q(G)$ denotes the space of homogeneous quasimorphisms and $\widetilde{QH}(G)$ the quotient of the space of all quasimorphisms by the subspace spanned by bounded functions and homomorphisms. We use four standard facts.

(Q1) (Homogenization, [5, Lemma 2.21].) If ψ is a quasimorphism, then for every $a \in G$ the limit $\bar{\psi}(a) = \lim_{m\to\infty} \psi(a^m)/m$ exists; moreover $\bar{\psi} \in Q(G)$ and $|\bar{\psi} - \psi| \leq D(\psi)$ pointwise.

(Q2) ([5, Ex. 2.16].) A quasimorphism has defect 0 if and only if it is a homomorphism.

(Q3) ([5, §2.2.3].) Homogeneous quasimorphisms are class functions, and $|\varphi([a,b])| \leq D(\varphi)$ for every commutator.

(Q4) (Bavard [1, Lemma 3.6]; [5, Lemma 2.24].) For $\varphi \in Q(G)$, $\sup_{a,b\in G} |\varphi([a,b])| = D(\varphi)$.

Lemma 2.1. *Let $\varphi \in Q(G)$ and let g be a product of $k \geq 1$ commutators in G. Then $|\varphi(g)| \leq (2k-1)D(\varphi)$.*

Proof. Induction on k; the case $k = 1$ is (Q3). If $g = g'[a_k, b_k]$ with g' a product of $k-1 \geq 1$ commutators, then by the defect inequality, the inductive hypothesis and (Q3), $|\varphi(g)| \leq |\varphi(g')| + |\varphi([a_k, b_k])| + D \leq (2k-3)D + D + D$. □

Lemma 2.2. *Let $\pi\colon G \to H$ be an epimorphism and $\varphi \in Q(H)$. Then $\varphi \circ \pi \in Q(G)$ and $D(\varphi \circ \pi) = D(\varphi)$.*

Proof. $\varphi(\pi(a)^m) = m\varphi(\pi a)$ gives homogeneity, and since $\{(\pi a, \pi b) : a, b \in G\} = H \times H$ the two suprema defining the defects run over the same set of values. □

Lemma 2.3. *If ψ is a quasimorphism on H whose class in $\widetilde{QH}(H)$ is nonzero, then its homogenization $\varphi = \bar{\psi}$ satisfies $D(\varphi) > 0$.*

Proof. By (Q1), $\psi - \varphi$ is bounded. If $D(\varphi) = 0$ then φ is a homomorphism by (Q2), so $\psi = \varphi + (\psi - \varphi)$ lies in the span of homomorphisms and bounded functions, i.e. its class in $\widetilde{QH}(H)$ is zero. □

2.2. **The deep input.**

Proposition 2.4 (Bestvina–Feighn; Osin, Bestvina–Bromberg–Fujiwara)**.** $\dim \widetilde{QH}(\mathrm{Out}\, F_n) = \infty$ *for every $n \geq 2$.*

This is [4, Cor. 4.30], which is printed without a lower bound on n (the construction uses fully irreducible automorphisms and the action of $\mathrm{Out}\, F_n$ on a hyperbolic $\mathrm{Out}\, F_n$-complex); the space $\widetilde{QH}$ there is defined exactly as above. Independently, Osin [11, §8(b)] shows that $\mathrm{Out}\, F_n$ is acylindrically hyperbolic for $n \geq 2$, using the same Bestvina–Feighn complex, and [2, Cor. 1.2] gives $\dim \widetilde{QH}(G) = \infty$ for every acylindrically hyperbolic G (their $\widetilde{QC}(G; \rho)$ with ρ the trivial representation on $\mathbb{R}$ is $\widetilde{QH}(G)$, and the fixed-vector hypothesis of that corollary is vacuous for trivial ρ). For $n \geq 3$ either source suffices; for $n = 2$ we rely on the second.

2.3. **Proof of Theorem 1.1.**

Proof of (a). By Proposition 2.4 there is a quasimorphism ψ on O_n whose class in $\widetilde{QH}(O_n)$ is nonzero; by Lemma 2.3 its homogenization $\varphi \in Q(O_n)$ has $D(\varphi) > 0$. Put $\Phi = \varphi \circ \pi$; by Lemma 2.2, $\Phi \in Q(A_n)$ with $D(\Phi) = D(\varphi) > 0$. By (Q4) there are $a, b \in A_n$ with $\Phi(c) \neq 0$ for $c = [a, b] \in A'_n$; put $\lambda = |\Phi(c)|/D(\Phi) > 0$. Fix $m \geq 1$. Since $\Phi(c^m) = m\Phi(c) \neq 0 = \Phi(1)$, we have $c^m \neq 1$, so $k = \mathrm{cl}_{A_n}(c^m)$ satisfies $1 \leq k < \infty$ ($c^m \in A'_n$). Lemma 2.1 gives $m\lambda D(\Phi) = |\Phi(c^m)| \leq (2k-1)D(\Phi)$, i.e. $\mathrm{cl}_{A_n}(c^m) \geq \frac{1}{2}(\lambda m + 1)$. As $c^m \in A'_n$ for all m, $k_n = \infty$. □

Lemma 2.5. *For $n \geq 3$, $\mathrm{SAut}\, F_n$ is perfect, $A'_n = \mathrm{SAut}\, F_n$, and $H_1(A_n) = \mathbb{Z}/2$.*

Proof. Let $x_1, \ldots, x_n$ be a basis of F_n and call $x_i^{\pm 1}$ letters (we use u, v, w for letters, to keep them apart from the commutator $c = [a, b]$ of Theorem 1.1). For letters $u \neq v^{\pm 1}$ let E_{uv} be the automorphism $u \mapsto uv$ fixing every letter other than $u^{\pm 1}$; automorphisms act on the right. By Gersten's theorem [7] (in the form reproduced in [12, Thm. 2.1]), $\mathrm{SAut}\, F_n$ is generated by the E_{uv} for $n \geq 3$. For letters u, v, w with $u \neq v^{\pm 1}$, $v \neq w^{\pm 1}$, $u \neq w^{\pm 1}$ one checks directly that $E_{uv} E_{vw} E_{uv}^{-1} E_{vw}^{-1} = E_{uw}$: the composite sends $u \mapsto uv \mapsto uvw \mapsto uv^{-1}vw = uw \mapsto uw$ and $v \mapsto v \mapsto vw \mapsto vw \mapsto vw^{-1}w = v$, and it fixes every letter other than $u^{\pm 1}, v^{\pm 1}$, since each factor does; so it agrees with E_{uw} on a basis. Given a generator E_{uw}, the letters u, w involve at most two of the $n \geq 3$ indices; choosing v a third basis element exhibits E_{uw} as a commutator of elements of $\mathrm{SAut}\, F_n$. Hence $\mathrm{SAut}\, F_n$ is perfect. Since det is onto ($x_1 \mapsto x_1^{-1}$ has determinant -1), $A_n/\mathrm{SAut}\, F_n \cong \{\pm 1\}$ is abelian, so $A'_n \leq \mathrm{SAut}\, F_n = (\mathrm{SAut}\, F_n)' \leq A'_n$. □

Proof of (b). By Lemma 2.5, $c^m \in A_n' = \mathrm{SAut}\, F_n$ has finite $\mathrm{cl}_{A_n'}(c^m)$, and a product of k commutators of elements of A_n' is a product of k commutators of elements of A_n, so $\mathrm{cl}_{A_n'}(c^m) \geq \mathrm{cl}_{A_n}(c^m)$. For O_n: $\varphi(\pi(c)^m) = \Phi(c^m) = m\Phi(c) \neq 0$, so $\pi(c)^m \neq 1$ and $k = \mathrm{cl}_{O_n}(\pi(c)^m) \geq 1$ is finite ($\pi(c) = [\pi a, \pi b] \in O_n'$); Lemma 2.1 applied to $\varphi \in Q(O_n)$, whose defect is $D(\Phi)$, gives $m\lambda D(\Phi) \leq (2k-1)D(\Phi)$, i.e. $\mathrm{cl}_{O_n}(\pi(c)^m) \geq \frac{1}{2}(\lambda m + 1)$. □

Second proof that $k_n = \infty$. Bestvina–Bromberg–Fujiwara [3, Thm. 1.1] prove: if G is acylindrically hyperbolic and w is a word with $d(w) \neq 1$, where $d(w)$ is the greatest common divisor of the exponent sums of w (with $d(w) = 0$ if all vanish), then the verbal subgroup $w(G)$ has infinite width. For $w = x_1 x_2 x_1^{-1} x_2^{-1}$ one has $d(w) = 0$, $w(G) = G'$, and the verbal length is the commutator length (all as stated in [3, §1]). Since $\mathrm{Out}\, F_n$ is acylindrically hyperbolic for $n \geq 2$ [11, §8(b)] (the definition in [3] is Osin's), cl_{O_n} is unbounded on O_n'. Finally $\pi(A_n') = O_n'$ and $\mathrm{cl}_{A_n}(g) \geq \mathrm{cl}_{O_n}(\pi g)$ for $g \in A_n'$ [5, Lemma 2.4], so cl_{A_n} is unbounded on A_n', i.e. $k_n = \infty$; and $\mathrm{cl}_{A_n'} \geq \mathrm{cl}_{A_n}$ on A_n' gives the same for $\mathrm{SAut}\, F_n$ when $n \geq 3$. □

Remark 2.6 ($n = 2$). Part (a) and its proof hold verbatim for $n = 2$, using the Osin–Bestvina–Bromberg–Fujiwara source of Proposition 2.4; likewise the Second proof. This recovers the notebook's $k_2 = \infty$. Lemma 2.5 needs $n \geq 3$: A_2' is a proper subgroup of $\mathrm{SAut}\, F_2$, since A_2 maps onto $\mathrm{GL}_2(\mathbb{Z})$, whose abelianization is $(\mathbb{Z}/2)^2$. So part (b) is stated for $n \geq 3$ only.

Remark 2.7 (Linear quotients cannot see this). The image of A_n' under $A_n \to \mathrm{GL}_n(\mathbb{Z})$ is $\mathrm{SL}_n(\mathbb{Z})$, whose commutator width is finite for $n \geq 3$ (Newman [10]: at most $c \log n + 40$ commutators for an absolute constant c). Hence no homogeneous quasimorphism of A_n that factors through $\mathrm{GL}_n(\mathbb{Z})$ can have positive defect (by Lemma 2.1 it would be bounded on the perfect group $\mathrm{SL}_n(\mathbb{Z})$, hence zero there by homogeneity, hence a homomorphism of $\mathrm{GL}_n(\mathbb{Z})$ by (Q4)), and any proof of Theorem 1.1 must use the non-linear structure of $\mathrm{Out}\, F_n$ — here, its actions on hyperbolic complexes. For the same reason $k_n = \infty$ cannot be detected by finite quotients of A_n or by computation in $\mathrm{GL}_n(\mathbb{Z})$.

3. Open questions

The proof is not effective: it produces no explicit c and no value of λ. Bestvina–Feighn's quasimorphisms are unbounded on the cyclic subgroup generated by a fully irreducible automorphism, so a lift $c \in A_n$ of a suitable power of a fully irreducible element should satisfy $\mathrm{scl}_{A_n}(c) > 0$ with a computable lower bound; we did not pursue this. Nothing seems to be known about the set of elements of A_n' on which scl_{A_n} vanishes, or about rationality of scl on $\mathrm{Aut}\, F_n$ or $\mathrm{Out}\, F_n$.

Methods and AI disclosure

This note was produced by an automated research pipeline built on a large language model (`claude-fable-5.1`), and its claims were verified before assembly: the claim package consisting of Theorem 1.1 with its first proof and Lemma 2.5 passed two independent adversarial verification passes by fresh-context instances shown only the problem statement and the candidate text, in which every lemma was re-derived and the relator identity of Lemma 2.5 was recomputed by hand and by machine, together with a two-stage citation audit that located every cited statement, recorded it with its hypotheses and the source's standing conventions, took bibliographic data from publisher or DOI records, and checked the published wording of [4, Cor. 4.30]. The Second proof and Remark 2.7 were added after that verification from statements covered by the same audit and received a further scoped verification pass. The facts of Lemma 2.5 were also corroborated in GAP 4.15.1 [13] for $n = 3, 4$, from two published presentations (each relator first checked to be an automorphism of F_n) and, independently, from GAP's own construction of $\mathrm{Aut}\, F_3$; these computations are not used in the proofs. Complete machine records — scripts, transcripts and verifier reports — are preserved in the repository accompanying this note and are available from the authors.

Every result used is cited where used; to our knowledge the only new content is the assembly and the observation that it answers Problem 14.15. Our searches found no published statement of the value of k_n, with the caveat that zbMATH and MathSciNet were not consulted.

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