AN ANSWER TO PROBLEM 14.23 FROM THE KOUROVKA NOTEBOOK

Abstract. Problem 14.23 of the Kourovka Notebook (O. V. Bogopolski, 1999) asks whether there is a recursive function $f\colon \mathbb{N} \to \mathbb{N}$ such that every automorphism α of the free group F_n admits a basis of its fixed subgroup $\mathrm{Fix}(\alpha)$ consisting of elements of length at most $f(\|\alpha\|)$, where $\|\alpha\|$ is the maximum length of the images of the fixed basis. If f may depend on n, the answer is yes; if one f must serve all n, the answer is no, already at the smallest possible norm. For every $r \geq 2$ we construct $\zeta_r \in \mathrm{Aut}\, F_{r+1}$ with $\|\zeta_r\| = \|\zeta_r^{-1}\| = 2$ whose fixed subgroup is infinite cyclic, generated by a primitive element of length $r+1$; so no function of $\|\alpha\|$ — recursive or not, or even of the pair $(\|\alpha\|, \|\alpha^{-1}\|)$ — can serve all n at once. Since every automorphism of norm 1 is a signed permutation of the basis, whose fixed subgroup is a free factor generated by basis letters, the threshold is sharp: the uniform bound exists at norm 1 and at no other norm, and the obstruction is combinatorial rather than computability-theoretic. A second family, of norm 3, turns out to be the Artin automorphism of a periodic braid, its fixed subgroup generated exactly by the boundary word. For each fixed n the answer is yes: the pointwise optimal bound $f^\circ(n, \cdot)$ is itself computable, by an explicit terminating procedure, and the existence of a recursive bound at a fixed n is equivalent to the computability of bases of fixed subgroups — established by Bogopolski–Maslakova (2016) and, independently, by Feighn–Handel (2018).

1. Introduction

Problem 14.23 of the Kourovka Notebook, posed by O. V. Bogopolski in the 14th issue (1999), reads as follows [9, Problem 14.23].

> **14.23.** Let F_n be a free group with basis $\{x_1, \ldots, x_n\}$, and let $|\cdot|$ be the length function with respect to this basis. For $\alpha \in \mathrm{Aut}\, F_n$ we put $\|\alpha\| = \max\{|\alpha(x_1)|, \ldots, |\alpha(x_n)|\}$. Is it true that there is a recursive function $f\colon \mathbb{N} \to \mathbb{N}$ with the following property: for any $\alpha \in \mathrm{Aut}\, F_n$ there is a basis $\{y_1, \ldots, y_k\}$ of $\mathrm{Fix}(\alpha) = \{x \mid \alpha(x) = x\}$ such that $|y_i| \leqslant f(\|\alpha\|)$ for all $i = 1, \ldots, k$?
>
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The problem carries no star and no editorial comment in the current edition of the Notebook, and it does not appear in the Archive of Solved Problems [9].

The statement has two readings, according as f may or may not depend on n: the *per-rank reading* — for every n there is a recursive $f_n\colon \mathbb{N} \to \mathbb{N}$ bounding Fix-basis lengths in terms of $\|\alpha\|$ over $\mathrm{Aut}\, F_n$ — and the *uniform reading*, in which a single f serves all n simultaneously. The two readings provably diverge, and both are settled below: the uniform reading fails, by an explicit norm-2 family (Theorem 1.2), and the per-rank reading holds, with the optimal bound itself computable (Theorem 1.4).

Remark 1.1 (on the parse). The sentence opens "Let F_n be a free group …", binding n before the existential quantifier over f, so the strictest parse allows f to depend on n; on the other hand the signature $f\colon \mathbb{N} \to \mathbb{N}$, in $\|\alpha\|$ alone, and the convention of the companion Problem 14.24 (same author, same norm, with $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and no dependence on n) suggest uniformity. We do not know which reading was intended, and neither is substituted for the other.

Notation. Fix a basis $X = \{x_1, \ldots, x_n\}$ of F_n; elements are reduced words over $X^{\pm}$, $|w|$ is reduced length, $|1| = 0$, and $\|\alpha\| := \max_i |\alpha(x_i)|$ (≥ 1: an automorphism kills no x_i). For $g \in F_n$, γ_g is the inner automorphism $u \mapsto gug^{-1}$, and $C(g) := \{u : ug = gu\}$. An endomorphism is determined by the n-tuple of reduced words $\vec{w}_\varphi = (\varphi(x_1), \ldots, \varphi(x_n))$, and every n-tuple $\vec{w}$ of reduced words determines an endomorphism $\varphi_{\vec{w}}\colon x_i \mapsto w_i$; the correspondence $\alpha \mapsto \vec{w}_\alpha$ is a norm-preserving bijection from $\mathrm{Aut}\, F_n$ onto the set of *automorphism tuples*. For fixed n, m the set $T(n, m)$ of n-tuples of reduced words of length at most m is

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finite and enumerable, and whether $\varphi_{\vec{w}}(u) = u$ for a given reduced u is decided by substituting, reducing, and comparing. For a finitely generated $H \le F_n$ let

$$M(H) := \min_{\text{bases } Y \text{ of } H} \max_{y \in Y} |y|, \qquad M(1) := 0,$$

the least L such that H has a basis of elements of length at most L (the empty set is the basis of the trivial subgroup, for which the length condition of 14.23 is vacuous). The *optimal bound function* is

$$f^{\circ}(n, m) := \max\{\, M(\mathrm{Fix}(\alpha)) : \alpha \in \mathrm{Aut}\, F_n,\ \|\alpha\| \le m \,\} \qquad (\max \emptyset := 0),$$

monotone nondecreasing in m since the tuple sets $T(n, m)$ are nested; here $M(\mathrm{Fix}(\alpha))$ is well defined because $\mathrm{Fix}(\alpha)$ is finitely generated — for us a consequence of the algorithms cited in Section 4, independent of the theorem of Bestvina–Handel [1] that $\mathrm{rk}\,\mathrm{Fix}(\alpha) \le n$. Put $F(m) := \sup_n f^{\circ}(n, m) \in \mathbb{N} \cup \{\infty\}$. A recursive function is a total computable function $\mathbb{N} \to \mathbb{N}$; for "algorithm" we use the informal Church–Turing convention — all procedures below are explicit finite searches.

Results. The uniform reading fails at the first opportunity, for combinatorial reasons.

Theorem 1.2 (the uniform reading fails, at norm two)**.** *For every $n \ge 3$ there is an automorphism $\zeta \in \mathrm{Aut}\, F_n$ with $\|\zeta\| = \|\zeta^{-1}\| = 2$ whose fixed subgroup is infinite cyclic, generated by a primitive element of length n, so that every basis of $\mathrm{Fix}(\zeta)$ consists of a single word of length exactly n. Hence $f^{\circ}(n, 2) \ge n$ and $F(2) = \infty$: no function $f : \mathbb{N} \to \mathbb{N}$ whatsoever — recursive or not, of $\|\alpha\|$ alone or of the pair $(\|\alpha\|, \|\alpha^{-1}\|)$ — has the property of Problem 14.23 uniformly in n, even restricted to automorphisms of norm 2. The two-norm variant fails at the single value $(2, 2)$.*

Corollary 1.3 (the threshold is sharp)**.** *$F(1) = 1$, and $F(m) = \infty$ for every $m \ge 2$. The uniform reading of Problem 14.23 fails exactly from the first nontrivial norm level on.*

The norm-1 evaluation is elementary: an automorphism of norm 1 is a signed permutation of the basis, and its fixed subgroup is the free factor generated by the fixed basis letters (Proposition 2.10). The case $m \ge 3$ follows from $m = 2$ by monotonicity, and was also proved directly by an independent norm-3 construction of a rather different flavour: the second family turns out to be the Artin automorphism of a periodic braid, with fixed subgroup generated exactly by the boundary word of the punctured disk (Section 3).

Theorem 1.4 (the per-rank reading holds)**.** *For every $n \ge 1$ there is a total recursive function $f_n : \mathbb{N} \to \mathbb{N}$ such that every $\alpha \in \mathrm{Aut}\, F_n$ has a basis $\{y_1, \dots, y_k\}$ of $\mathrm{Fix}(\alpha)$ with $|y_i| \le f_n(\|\alpha\|)$ for all i. One may take $f_n = f^{\circ}(n, \cdot)$, the pointwise optimal monotone bound, and this function is computed by an explicit terminating procedure.*

Theorem 1.5 (the equivalence)**.** *Fix $n \ge 1$. The following are equivalent.*

1. *Some recursive $f_n : \mathbb{N} \to \mathbb{N}$ has the property of Problem 14.23 for $\mathrm{Aut}\, F_n$.*
2. *There is a total algorithm that, given (the images of the basis under) an automorphism $\alpha \in \mathrm{Aut}\, F_n$, outputs a basis of $\mathrm{Fix}(\alpha)$.*

Remark 1.6 (history). Statement (2) is precisely the existence half of "Problem A" of Bogopolski–Maslakova [2] — "Find an efficient algorithm for computing a basis of $\mathrm{Fix}(\phi)$" — open, by their account, for almost twenty years before it was solved in [2] and again, independently, in [6]. So the per-rank reading of Problem 14.23, posed in 1999, was equivalent all along to a computability problem that its proposer himself later solved. In the literature searches performed for this note (described in the final section) we found no published statement of this equivalence, no mention of Problem 14.23 alongside [2] or [6], and no printed length-bound corollary to either paper; the elementary bridge of Section 4 appears to be the missing link — which would also explain why the problem could remain formally open in the Notebook after 2016 under the per-rank reading, while under the uniform reading it was, until now, genuinely open.

Quantitatively, the per-rank bounds are constrained from below: inner automorphisms force $f(2\ell + 1) \ge \ell$ for every valid f and every $n \ge 2$ (Theorem 4.5), and machine-certified values of the optimal bound begin

$$f^{\circ}(2,1) = 1, \quad f^{\circ}(2,2) = 4, \quad f^{\circ}(2,3) = 4, \quad f^{\circ}(2,4) = 16, \quad f^{\circ}(3,2) \ge 6, \quad f^{\circ}(4,2) \ge 22$$

(Section 5). No explicit upper bound — primitive recursive or otherwise — for any f_n appears to be extractable from the published literature (Remark 4.4).

Section 2 constructs ζ_r and proves Theorem 1.2 and Corollary 1.3; Section 3 treats the norm-three family, its braid provenance, and a contrasting norm-two family whose fixed subgroup does not grow; Section 4 proves Theorems 1.4 and 1.5 and the lower bounds; Section 5 reports the certified computations; Section 6 collects open questions. This note was produced by an automated pipeline built on a large language model and verified independently of its authorship; provenance and verification are stated in the final, unnumbered section.

2. The norm-two family and the sharp threshold

In this section it is convenient to index the basis of F_n cyclically: we write F_{r+1} ($r \geq 2$) for the free group on $z_0, z_1, \ldots, z_r$, with $n = r+1$; the z_i play the role of the x_i of Section 1. Set

$$P_j := z_0 z_1 \cdots z_{j-1} \quad (0 \leq j \leq r+1;\ P_0 := 1), \qquad Z := P_{r+1} = z_0 z_1 \cdots z_r.$$

We will use once and for all the classical structure of centralizers; since the proof is short we include it, and cite [10, Ch. I.2], [11, §4.4] for context. Subgroups of free groups are free (Nielsen–Schreier [11, Cor. 2.9]), and free groups are torsion-free (every element is conjugate to a cyclically reduced one, whose powers are reduced as written).

Lemma 2.1 (centralizers)**.** *Let F be free and $1 \neq g \in F$. Then $C_F(g)$ is infinite cyclic.*

Proof. (i) *Two commuting elements of F lie in a common cyclic subgroup*: if u, v commute, $\langle u, v\rangle$ is abelian and free, hence of rank at most 1 — a free group with two distinct basis elements a, b is nonabelian, $ab \neq ba$ being distinct reduced words. (ii) *A free group H with a nontrivial central element z is cyclic*: if $x \neq y$ are members of a basis B of H, then by (i) $\langle z, x\rangle = \langle c\rangle$ with $x = c^b$, $z = c^a$; abelianizing (H^{ab} free abelian on B), the e_x-coordinate of $e_x = b\,[c]$ gives $b = \pm 1$, so $c = x^{\pm 1}$ and $z \in \langle x\rangle$; symmetrically $z \in \langle y\rangle$; but $\langle x\rangle \cap \langle y\rangle = 1$ for distinct basis elements, so $z = 1$, a contradiction. Hence $|B| \leq 1$, and $z \neq 1$ makes H infinite cyclic. (iii) $C_F(g)$ is free with $1 \neq g \in Z(C_F(g))$; apply (ii). □

2.1. **The family.** For $r \geq 2$ define $\zeta = \zeta_r$ on F_{r+1} by

$$\zeta(z_0) = z_0 z_1, \qquad \zeta(z_j) = z_{j+1} \quad (1 \leq j \leq r-2), \qquad \zeta(z_{r-1}) = z_r z_0^{-1}, \qquad \zeta(z_r) = z_0.$$

(For $r = 2$ the middle range is empty: $\zeta = (z_0 z_1,\ z_2 z_0^{-1},\ z_0)$.)

Lemma 2.2 (automorphism; both norms two)**.** *$\zeta \in \operatorname{Aut} F_{r+1}$, and the map*

$$\xi(z_0) = z_r, \qquad \xi(z_1) = z_r^{-1} z_0, \qquad \xi(z_j) = z_{j-1} \quad (2 \leq j \leq r-1), \qquad \xi(z_r) = z_{r-1} z_r$$

is its inverse. $\|\zeta\| = \|\zeta^{-1}\| = 2$ exactly.

Proof. $\xi \circ \zeta = \mathrm{id}$ on the basis: $\xi(\zeta(z_0)) = \xi(z_0 z_1) = z_r \cdot z_r^{-1} z_0 = z_0$; for $1 \leq j \leq r-2$, $\xi(\zeta(z_j)) = \xi(z_{j+1}) = z_j$ ($2 \leq j+1 \leq r-1$); $\xi(\zeta(z_{r-1})) = \xi(z_r z_0^{-1}) = z_{r-1} z_r \cdot z_r^{-1} = z_{r-1}$; $\xi(\zeta(z_r)) = \xi(z_0) = z_r$. $\zeta \circ \xi = \mathrm{id}$ likewise: $\zeta(\xi(z_0)) = \zeta(z_r) = z_0$; $\zeta(\xi(z_1)) = \zeta(z_r^{-1} z_0) = z_0^{-1} \cdot z_0 z_1 = z_1$; for $2 \leq j \leq r-1$, $\zeta(\xi(z_j)) = \zeta(z_{j-1}) = z_j$ ($1 \leq j-1 \leq r-2$); $\zeta(\xi(z_r)) = \zeta(z_{r-1} z_r) = z_r z_0^{-1} \cdot z_0 = z_r$. So ζ is bijective with $\zeta^{-1} = \xi$. All eight displayed images are reduced of length at most 2, and both maps have an image of length exactly 2 ($r \geq 2$), so both norms equal 2. □

2.2. **The power identity.** We show $\zeta^{2r} = \gamma_Z$. Every displayed equality below is an identity between elements of the free group obtained by concatenation and the relations $zz^{-1} = 1$ alone; no normal-form or cancellation analysis is needed anywhere in this subsection.

Lemma 2.3 (prefix images)**.** *$\zeta(P_t) = P_{t+1}$ for $1 \leq t \leq r-1$. (Not for $t = 0$ or $t = r$: $\zeta(P_0) = 1 \neq P_1$, and $\zeta(P_r) = Z P_1^{-1}$ is computed inside Lemma 2.5.)*

Proof. $\zeta(P_t) = \zeta(z_0)\zeta(z_1)\cdots\zeta(z_{t-1}) = (z_0 z_1) \cdot z_2 \cdot z_3 \cdots z_t = P_{t+1}$, using $\zeta(z_0) = z_0 z_1$ and $\zeta(z_j) = z_{j+1}$ for $1 \leq j \leq t-1 \leq r-2$. □

Lemma 2.4 (power table)**.** *For $1 \le m \le r-1$:*

$$\begin{aligned}
\zeta^m(z_0) &= P_{m+1}; \\
\zeta^m(z_j) &= z_{j+m} && (1 \le j \le r-1-m); \\
\zeta^m(z_{r-m}) &= z_r P_1^{-1} \ (= z_r z_0^{-1}); \\
\zeta^m(z_j) &= P_t z_t^{-1} P_t^{-1} \quad \textit{with } t := j-r+m && (r-m+1 \le j \le r-1); \\
\zeta^m(z_r) &= P_m.
\end{aligned}$$

Proof. Induction on m. The case $m=1$ is the definition of ζ (ranges: middle shift range $1,\dots,r-2$, seam at $j=r-1$, fourth line's range empty, $z_r \mapsto z_0 = P_1$, $z_0 \mapsto z_0 z_1 = P_2$). For $m \to m+1$ ($m \le r-2$), using $\zeta^{m+1}(z) = \zeta(\zeta^m(z))$:

- $1 \le j \le r-1-(m+1)$: $\zeta(z_{j+m}) = z_{j+m+1}$ since $1 \le j+m \le r-2$.
- $j = r-(m+1)$: here $j+m = r-1$ lies in the m-table's shift range ($j = r-1-m \ge 1$), so $\zeta^{m+1}(z_j) = \zeta(z_{r-1}) = z_r z_0^{-1}$ (the new seam).
- $j = r-m$ (the old seam): $\zeta(z_r P_1^{-1}) = \zeta(z_r)\cdot\zeta(z_0)^{-1} = z_0 \cdot (z_0 z_1)^{-1} = z_0 z_1^{-1} z_0^{-1} = P_1 z_1^{-1} P_1^{-1}$, which is the fourth line's form with $t = (r-m)-r+(m+1) = 1$.
- $r-m+1 \le j \le r-1$ (the old fourth line, $t = j-r+m \in [1, m-1] \subseteq [1, r-3]$): $\zeta(P_t z_t^{-1} P_t^{-1}) = \zeta(P_t)\cdot\zeta(z_t)^{-1}\cdot\zeta(P_t)^{-1} = P_{t+1} z_{t+1}^{-1} P_{t+1}^{-1}$ by Lemma 2.3 ($t \le r-1$) and $\zeta(z_t) = z_{t+1}$ ($t \le r-2$); and $t+1 = j-r+(m+1)$.
- z_r: $\zeta(P_m) = P_{m+1}$ by Lemma 2.3 ($m \le r-1$). z_0: $\zeta(P_{m+1}) = P_{m+2}$ by Lemma 2.3 ($m+1 \le r-1$). □

Lemma 2.5 (the half power $\eta := \zeta^r$)**.**

$$\eta(z_0) = Z P_1^{-1}; \qquad \eta(z_j) = P_j z_j^{-1} P_j^{-1} \quad (1 \le j \le r-1); \qquad \eta(z_r) = P_r.$$

Proof. Apply ζ to the $m = r-1$ table (whose shift range is empty and whose seam sits at $j=1$):

- z_0: $\zeta(P_r) = \zeta(z_0)\cdots\zeta(z_{r-1}) = (z_0 z_1)\cdot z_2 \cdots z_{r-1} \cdot (z_r z_0^{-1}) = P_r z_r z_0^{-1} = Z z_0^{-1} = Z P_1^{-1}$.
- $j=1$ (the old seam): $\zeta(z_r P_1^{-1}) = z_0 (z_0 z_1)^{-1} = P_1 z_1^{-1} P_1^{-1}$.
- $2 \le j \le r-1$ (the old fourth line, $t = j-1 \le r-2$): $\zeta(P_{j-1} z_{j-1}^{-1} P_{j-1}^{-1}) = P_j z_j^{-1} P_j^{-1}$ (Lemma 2.3 and the shift, as in Lemma 2.4).
- z_r: $\zeta(P_{r-1}) = P_r$ (Lemma 2.3). □

Lemma 2.6 (the square)**.** $\eta^2 = \gamma_Z$*; hence* $\zeta^{2r} = \gamma_Z$*.*

Proof. First, $\eta(P_j) = Z P_j^{-1}$ for $1 \le j \le r$, by induction on j: $\eta(P_1) = \eta(z_0) = Z P_1^{-1}$; and for $j \le r-1$, $\eta(P_{j+1}) = \eta(P_j)\cdot\eta(z_j) = Z P_j^{-1} \cdot P_j z_j^{-1} P_j^{-1} = Z z_j^{-1} P_j^{-1} = Z (P_j z_j)^{-1} = Z P_{j+1}^{-1}$. Consequently $\eta(Z) = \eta(P_r)\cdot\eta(z_r) = Z P_r^{-1} \cdot P_r = Z$. Now:

- $\eta^2(z_0) = \eta(Z P_1^{-1}) = \eta(Z)\cdot\eta(P_1)^{-1} = Z \cdot (Z P_1^{-1})^{-1} = Z P_1 Z^{-1} = Z z_0 Z^{-1}$.
- $\eta^2(z_j) = \eta(P_j)\cdot\eta(z_j)^{-1}\cdot\eta(P_j)^{-1} = Z P_j^{-1} \cdot (P_j z_j P_j^{-1}) \cdot P_j Z^{-1} = Z z_j Z^{-1}$ ($1 \le j \le r-1$).
- $\eta^2(z_r) = \eta(P_r) = Z P_r^{-1}$, and this equals $Z z_r Z^{-1}$: indeed $Z P_r^{-1} = (P_r z_r) P_r^{-1} = P_r z_r P_r^{-1}$, while $Z z_r Z^{-1} = (P_r z_r)\cdot z_r \cdot (P_r z_r)^{-1} = P_r z_r \cdot z_r \cdot z_r^{-1} P_r^{-1} = P_r z_r P_r^{-1}$.

So η^2 agrees with γ_Z on a basis, and $\zeta^{2r} = \eta^2 = \gamma_Z$. □

2.3. **The fixed subgroup, exactly.**

Lemma 2.7 (Z is primitive; its centralizer)**.** $\{Z, z_1, \dots, z_r\}$ *is a basis of F_{r+1}; in particular $Z \ne 1$, $|Z| = r+1 = n$ (Z is a positive word, so reduced), and $C(Z) = \langle Z \rangle$.*

Proof. The endomorphism fixing $z_1,\dots,z_r$ and sending $z_0 \mapsto Z$ is surjective ($z_0 = Z\cdot(z_1\cdots z_r)^{-1}$ lies in the image), hence an automorphism (F_n is Hopfian [11, Thm. 2.13]); it carries the standard basis to $\{Z, z_1, \dots, z_r\}$, which is therefore a basis. Next, a basis element is its own maximal root: if $Z = u^d$ with $d \ge 2$ then in $H_1 = \mathbb{Z}^n$ the class of Z, a member of a $\mathbb{Z}$-basis of H_1 (automorphisms of F_n induce automorphisms of H_1 carrying basis to basis), would equal $d\cdot$(class of u), contradicting indivisibility of basis vectors. Writing $C(Z) = \langle c \rangle$ (Lemma 2.1) and $Z = c^s$ ($s \ne 0$ since $Z \ne 1$), rootlessness forces $s = \pm 1$, so $C(Z) = \langle Z \rangle$. □

Theorem 2.8. *For every $r \ge 2$, with $n = r+1$:* $\mathrm{Fix}(\zeta_r) = \langle Z \rangle$*, infinite cyclic. Its only bases are $\{Z\}$ and $\{Z^{-1}\}$, of length exactly n. And $\|\zeta_r\| = \|\zeta_r^{-1}\| = 2$.*

Proof. Z is fixed: $\zeta(Z) = \zeta(z_0)\zeta(z_1)\cdots\zeta(z_{r-1})\zeta(z_r) = (z_0z_1)\cdot z_2\cdots z_{r-1}\cdot(z_rz_0^{-1})\cdot z_0 = Z$. Conversely if $\zeta(w) = w$ then $\zeta^{2r}(w) = w$, i.e. $ZwZ^{-1} = w$ (Lemma 2.6), i.e. $w \in C(Z) = \langle Z\rangle$ (Lemma 2.7). So $\mathrm{Fix}(\zeta) = \langle Z\rangle$ with $Z \neq 1$; the bases of an infinite cyclic group are its two generators, i.e. $\{Z^{\pm1}\}$, and $|Z^{\pm1}| = n$ (Lemma 2.7). Norms: Lemma 2.2. □

The mechanism in the last proof has no hidden nontriviality condition; we record it in general, for use in Section 5.

Lemma 2.9 (inner powers pin the fixed subgroup)**.** *Let $n \geq 2$, $\alpha \in \mathrm{Aut}\, F_n$, $k \geq 1$ and $1 \neq g \in F_n$ with $\alpha^k = \gamma_g$. Then $\alpha(g) = g$, and $\mathrm{Fix}(\alpha) = \langle\hat{g}\rangle$, where $\hat{g}$ is the maximal root of g: the unique root-free u with $u^s = g$ for some $s \geq 1$.*

Proof. $\gamma_{\alpha(g)} = \alpha\gamma_g\alpha^{-1} = \alpha\cdot\alpha^k\cdot\alpha^{-1} = \gamma_g$, and the center of F_n is trivial for $n \geq 2$ (a free group with nontrivial center is cyclic — step (ii) of the proof of Lemma 2.1 — and F_n is not, as $x_1x_2 \neq x_2x_1$), so $\alpha(g) = g$. By Lemma 2.1 write $C(g) = \langle c\rangle$ and $g = c^s$; replacing c by c^{-1} if necessary, $s \geq 1$. Then c is root-free (if $c = v^e$ then v commutes with c, hence with g, so $v \in \langle c\rangle$, $v = c^j$, $c = c^{je}$, $je = 1$, $e = 1$), and it is the unique root-free root of g with positive exponent: any root u of g commutes with g, so $u = c^i$, and root-freeness forces $i = \pm1$, while a negative exponent is excluded by $s \geq 1$. So $\hat{g} = c$ is well defined. Now $\mathrm{Fix}(\alpha) \subseteq \mathrm{Fix}(\alpha^k) = \mathrm{Fix}(\gamma_g) = C(g) = \langle\hat{g}\rangle$; and $\alpha(\hat{g})$ is again a root-free root of $\alpha(g) = g$ with positive exponent (automorphisms preserve root-freeness), so $\alpha(\hat{g}) = \hat{g}$. Hence $\langle\hat{g}\rangle \subseteq \mathrm{Fix}(\alpha) \subseteq \langle\hat{g}\rangle$. □

2.4. **The sharp threshold.**

Proof of Theorem 1.2. Take $\alpha = \zeta_{n-1}$ ($n \geq 3$). By Theorem 2.8, $\mathrm{Fix}(\alpha)$ is infinite cyclic on the primitive word Z of length n, and every basis of it is $\{Z^{\pm1}\}$; so $M(\mathrm{Fix}(\alpha)) = n$ and $f^\circ(n,2) \geq n$, whence $F(2) = \sup_n f^\circ(n,2) = \infty$. A uniform f would need $f(2) \geq n$ for all n; and since $\|\zeta_{n-1}^{-1}\| = 2$ as well, the two-norm variant $f(\|\alpha\|, \|\alpha^{-1}\|)$ fails equally, at the single value $(2,2)$. □

Proposition 2.10 (norm one: signed permutations)**.** *For every $n \geq 1$: an automorphism $\alpha \in \mathrm{Aut}\, F_n$ has $\|\alpha\| = 1$ if and only if it is a* signed permutation*, $\alpha(x_i) = x_{\sigma(i)}^{\varepsilon_i}$ with $\sigma \in \mathrm{Sym}(n)$ and $\varepsilon_i \in \{\pm1\}$; for such α, $\mathrm{Fix}(\alpha)$ is the free factor generated by $\{x_i : \sigma(i) = i,\ \varepsilon_i = +1\}$, so $M(\mathrm{Fix}(\alpha)) \leq 1$. Consequently $f^\circ(n,1) = 1$ for every n, and the uniform reading holds at $m = 1$ with $f(1) = 1$.*

Proof. If $\|\alpha\| = 1$, each $\alpha(x_i)$ is a single letter $x_{\sigma(i)}^{\varepsilon_i}$; were σ not surjective, say $j \notin \mathrm{im}\,\sigma$, the image of α would lie in the subgroup generated by the letters $\neq x_j^{\pm1}$, whose abelianized image misses e_j — so α would not be onto. Hence $\sigma \in \mathrm{Sym}(n)$; conversely a signed permutation has the obvious inverse signed permutation. Such an α permutes the alphabet $X^\pm$ injectively, so it sends a reduced word $w = a_1\cdots a_r$ to the reduced word $\alpha(a_1)\cdots\alpha(a_r)$ of the same length (a cancellation $\alpha(a_{j+1}) = \alpha(a_j)^{-1} = \alpha(a_j^{-1})$ would force $a_{j+1} = a_j^{-1}$); hence $\alpha(w) = w$ iff every letter of w is fixed, and a letter $x_i^{\pm1}$ is fixed iff $\sigma(i) = i$, $\varepsilon_i = +1$. So $\mathrm{Fix}(\alpha)$ is the free factor spanned by the fixed basis letters, and $M(\mathrm{Fix}(\alpha)) \leq 1$. Finally $f^\circ(n,1) = 1$: every norm-1 automorphism has $M(\mathrm{Fix}) \leq 1$, and $M(\mathrm{Fix}(\mathrm{id})) = M(F_n) = 1$ (a generating set inside the length-0 ball $\{1\}$ generates $1 \neq F_n$). □

Proof of Corollary 1.3. $F(1) = 1$ by Proposition 2.10. $F(2) = \infty$ by Theorem 1.2, and for $m \geq 2$ monotonicity gives $f^\circ(n,m) \geq f^\circ(n,2) \geq n$ for $n \geq 3$, so $F(m) = \infty$. (The case $m = 3$ is also proved directly, by an independent construction, in Section 3.) □

Two remarks locate the phenomenon. First, at fixed norm the *number* of basis elements is unbounded too, so no finite "universal basis list" can exist:

Proposition 2.11 (the number of basis elements grows)**.** *For $n \geq 2$ let $\alpha \in \mathrm{Aut}\, F_n$ be defined by $x_1 \mapsto x_1$, $x_i \mapsto x_ix_1$ ($i \geq 2$) — an automorphism of norm 2, with inverse $x_1 \mapsto x_1$, $x_i \mapsto x_ix_1^{-1}$. Then $\mathrm{rk}\,\mathrm{Fix}(\alpha) \geq n-1$.*

Proof. The $n-1$ elements x_1 and $x_2x_i^{-1}$ ($i = 3,\dots,n$) are fixed ($\alpha(x_2x_i^{-1}) = x_2x_1\cdot x_1^{-1}x_i^{-1} = x_2x_i^{-1}$), and their abelianized images e_1, $e_2 - e_i$ are $\mathbb{Z}$-linearly independent; a generating set of $\mathrm{Fix}(\alpha)$ of size k generates an abelian image containing $n-1$ independent vectors, so $k \geq n-1$. □

(By [1], $\operatorname{rk}\operatorname{Fix}(\alpha) \le n$ always; computations suggest the rank here is exactly n, confirmed for $n \le 5$, but this is not needed.) Problem 14.23 bounds lengths only, so this is consistent — the refutation had to come through lengths, as it did. Second, not every norm-2 family works: the shift-append family of Proposition 3.5 has $\|\alpha_n\| = 2$ but a fixed subgroup of rank 2 with $M = 3$, independently of n — and its inverse norm is n, which is why refutations must control both norms. Finally, the enumeration method of Theorem 1.4 dies structurally at the uniform reading: for fixed m the automorphism tuples range over all n, an infinite set, and with the finite enumeration goes the whole argument; by Theorem 1.5 the per-rank problem is a computability statement, while uniformity is a genuinely quantitative one.

Remark 2.12 (where the family comes from). ζ_r is the algebraic shadow of a fixed-point-free graph rotation; the proofs above stand alone, and this remark is motivation only. Let Γ_r be the graph with vertex set $\mathbb{Z}_{2r}$, cycle edges $e_i\colon i \to i+1$ ($i \in \mathbb{Z}_{2r}$) and diameters $d_i\colon i \to i+r$ ($0 \le i \le r-1$), and let φ be the rotation $+1$ (of order $2r$; it inverts each diameter after r steps and fixes no vertex). Take the spanning tree $T = \{e_0, \dots, e_{r-2}\} \cup \{d_0, \dots, d_{r-1}\}$ — the line $0, \dots, r-1$ plus all diameters — base vertex 0, and the drag path $w = e_0$. The non-tree edges are $e_{r-1}, e_r, \dots, e_{2r-1}$, giving the basis $z_0, \dots, z_r$ ($z_t \leftrightarrow$ crossing e_{r-1+t}), and $p \mapsto w \cdot \varphi(p) \cdot w^{-1}$ realizes ζ_r; the m-th power drags along $e_0 e_1 \cdots e_{m-1}$, and after $2r$ steps the drag path closes up into the full cycle, whose non-tree crossing word is $z_0 z_1 \cdots z_r = Z$: hence $\zeta^{2r} = \gamma_Z$. The mixed tree is the whole trick: it spreads the invariant cycle across *all* n basis elements (so the holonomy word has length n) while the rotation still moves each basis edge to an adjacent one (so the norm is 2). With the naive line tree the same outer class has a norm-2 representative whose fixed word is a single letter — the tree, not the class, is what makes the fixed word long. (This also explains why the holonomy mechanism of Section 3 stalls at norm 2: Lemma 3.1 needs a finite-order $\sigma \in \operatorname{Aut} F_n$ — a rotation fixing a vertex — while ζ_r has infinite order in $\operatorname{Aut} F_n$, since $\zeta_r^{2rt} = \gamma_{Z^t} \ne \mathrm{id}$ for $t \ne 0$.)

The family ζ_r is not extremal for its rank: the certified values of Section 5 give $f^\circ(3,2) \ge 6$ and $f^\circ(4,2) \ge 22$, against fixed generators of lengths 3 and 4 for ζ_2, ζ_3. The point of ζ_r is not per-rank optimality but a proof valid for all n simultaneously. All assertions about ζ_r were additionally verified by computer for $r \le 12$ (independently in a second toolchain for $r \le 6$ and $r = 12$), including exhaustive enumerations of short fixed words; the identity $\zeta^{2r} = \gamma_Z$ was further checked symbolically for $r \le 60$ (Section 5 and the final section).

3. The norm-three family and its braid provenance

We return to the notation F_n free on $x_1, \dots, x_n$. The family of this section refutes the uniform reading at norm $(3,3)$, independently of Section 2, by the holonomy of a twisted finite-order automorphism.

Lemma 3.1 (holonomy of a twisted finite-order automorphism)**.** *Let F be a free group, $\sigma \in \operatorname{Aut} F$ with $\sigma^N = \mathrm{id}$, $c \in F$, and $\beta := \gamma_c \circ \sigma$. Put $C := c \cdot \sigma(c) \cdot \sigma^2(c) \cdots \sigma^{N-1}(c)$. Then:*

(a) *$\beta(C) = C$. — Indeed $\sigma(C) = \sigma(c) \cdots \sigma^{N-1}(c) \cdot \sigma^N(c) = c^{-1} C c$, so $\beta(C) = c \cdot \sigma(C) \cdot c^{-1} = C$.*

(b) *$\operatorname{Fix}(\beta) \subseteq C_F(C)$. — If $u = c \cdot \sigma(u) \cdot c^{-1}$, substituting the equation into itself j times gives $u = [c\,\sigma(c) \cdots \sigma^{j-1}(c)]\ \sigma^j(u)\ [c\,\sigma(c) \cdots \sigma^{j-1}(c)]^{-1}$; at $j = N$: $u = CuC^{-1}$.*

(c) *If C is* primitive *(a member of some basis of F), then $\operatorname{Fix}(\beta) = \langle C \rangle$: $C_F(C)$ is infinite cyclic (Lemma 2.1), say $C_F(C) = \langle c_0 \rangle$ with $C = c_0^k$; a primitive element of F_n abelianizes to a member of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, whose content 1 must be divisible by $|k|$, forcing $|k| = 1$ — so $C_F(C) = \langle C \rangle$; combine with (a) and (b).*

Theorem 3.2. *For $n \ge 2$ let σ_n be the cyclic basis permutation $x_i \mapsto x_{i+1}$ (indices mod n) and $\beta_n := \gamma_{x_n} \circ \sigma_n$, explicitly*

$$\beta_n(x_i) = x_n x_{i+1} x_n^{-1}\ (1 \le i \le n-2), \qquad \beta_n(x_{n-1}) = x_n, \qquad \beta_n(x_n) = x_n x_1 x_n^{-1}.$$

Then:

(i) *$\beta_n \in \operatorname{Aut} F_n$ with $\|\beta_n\| = 3$ and $\|\beta_n^{-1}\| = 3$ ($\beta_n^{-1} = \gamma_{x_{n-1}^{-1}} \circ \sigma_n^{-1}$: $x_1 \mapsto x_{n-1}^{-1} x_n x_{n-1}$, $x_i \mapsto x_{n-1}^{-1} x_{i-1} x_{n-1}$ ($2 \le i \le n-1$), $x_n \mapsto x_{n-1}$);*

(ii) *$C := x_n x_1 x_2 \cdots x_{n-1}$ is primitive of length n (the endomorphism ψ fixing $x_1, \dots, x_{n-1}$ and sending $x_n \mapsto C$ has the explicit two-sided inverse $x_n \mapsto x_n (x_1 \cdots x_{n-1})^{-1}$ — a transvection — so $\psi \in \operatorname{Aut} F_n$ and $\{x_1, \dots, x_{n-1}, C\} = \psi(\{x_1, \dots, x_n\})$ is a basis);*

(iii) *$\operatorname{Fix}(\beta_n) = \langle C \rangle$ — infinite cyclic. Its only bases are $\{C\}$ and $\{C^{-1}\}$, of length exactly n.*

Proof. (i) composition of automorphisms; norms by inspection and by the displayed inverse (verified: β_n applied to each claimed preimage returns x_i, for all i). (ii) as stated. (iii) Lemma 3.1 with $N = n$, $c = x_n$: $\sigma_n^j(x_n) = x_j$ gives $C = c \cdot \sigma(c) \cdots \sigma^{n-1}(c) = x_n x_1 \cdots x_{n-1}$, so $\mathrm{Fix}(\beta_n) = \langle C \rangle$ by (c). A basis of an infinite cyclic group is a single generator, i.e. $C^{\pm 1}$; C is cyclically reduced, so $|C^{\pm 1}| = n$. □

Corollary 3.3. *$f^\circ(n, 3) \geq n$ for every $n \geq 2$, hence $F(3) = \infty$; and since $\|\beta_n^{-1}\| = 3$ as well, the two-norm variant $f(\|\alpha\|, \|\alpha^{-1}\|)$ fails independently at the value $(3, 3)$. Any true uniform-flavoured bound must involve n — and then $f^\circ(n, \cdot)$ is already computable (Theorem 1.4).* □

The contrast with the inner lower bound (Theorem 4.5) deserves note: conjugations give linear growth *along* m, at cost norm $2\ell + 1$ per length ℓ; the twisted cycle fixes the norm at 3 and grows the basis length with the *rank*. The assertions of Theorem 3.2 were verified by computer for $n \leq 6$ by exhaustive enumeration (inverse norms for $n \leq 8$); see Section 5.

Remark 3.4 (β_n is the Artin automorphism of a periodic braid). Recall (see [7]) that the braid group B_n acts on $F_n = \pi_1(D_n)$, free on the loops $y_1, \ldots, y_n$ around the punctures of the n-punctured disk, and that Artin's theorem [7, Thm. 1.3] characterizes braid automorphisms by two conditions: (1) every $f(y_i)$ is a conjugate of some y_j, and (2) $f(y_1 \cdots y_n) = y_1 \cdots y_n$, the product corresponding to the boundary of D_n. Under the relabeling $y_1 := x_n$, $y_j := x_{j-1}$ $(2 \leq j \leq n)$, the family of Theorem 3.2 satisfies both on the nose:

$$\beta_n(y_1) = y_1 y_2 y_1^{-1}, \qquad \beta_n(y_j) = y_1 y_{j+1} y_1^{-1} \ (2 \leq j \leq n-1), \qquad \beta_n(y_n) = y_1,$$

and $\beta_n(y_1 \cdots y_n) = \beta_n(C) = C$. So β_n is the Artin automorphism of a braid whose underlying permutation is an n-cycle — a periodic braid — the fixed word C is the boundary word of the punctured disk, and $\beta_n^n = \gamma_C$ (Lemma 3.1) instantiates the classical relation $\delta^n = \Delta^2$ [7]. What is new here, stated precisely: the *exact equality* $\mathrm{Fix}(\beta_n) = \langle C \rangle$ — Artin's theory gives the inclusion $\supseteq$, his condition (2), and our searches located no printed statement of the equality — and the connection to norm bounds and to Problem 14.23, for which we found no printed antecedent at all. In the other direction, the norm-2 family ζ_r *fails* Artin's condition (1): $\zeta_r(z_0) = z_0 z_1$ is conjugate to no generator (two nonzero abelianized coordinates) — and that is exactly what permits $\|\zeta_r\| = 2$, where conjugate-shaped images force norm at least 3: for $\beta = \gamma_c \circ \sigma$ with σ a signed permutation, the images $c \cdot \sigma(x_i) \cdot c^{-1}$ have odd reduced length. The two families are genuinely different.

3.1. **A norm-two family whose fixed subgroup does not grow.** Not every natural norm-2 family refutes uniformity; the following one conspicuously does not, and it also shows that the inverse norm is unbounded on automorphisms of norm 2 — so no approach to the uniform reading may pass through inverse-norm bounds, and the families above had to be built with *both* norms controlled.

Proposition 3.5 (shift-append)**.** *For $n \geq 2$ define $\alpha_n \in \mathrm{Aut}\, F_n$ by $\alpha_n(x_1) = x_1$, $\alpha_n(x_i) = x_i x_{i-1}$ $(2 \leq i \leq n)$. Then:*

(a) *$\|\alpha_n\| = 2$ while $\|\alpha_n^{-1}\| = n$;*
(b) *$\mathrm{Fix}(\alpha_n) = \langle x_1,\ x_2 x_1 x_2^{-1} \rangle$, free of rank 2 independently of n, with $M(\mathrm{Fix}(\alpha_n)) = 3$.*

In particular the shift-append family contributes only the n-independent bound $f^\circ(n, 2) \geq 3$.

Proof. (a) Define β by $\beta(x_1) = x_1$ and, recursively, $\beta(x_i) = x_i \cdot \beta(x_{i-1})^{-1}$. Then $\alpha_n \beta(x_i) = x_i x_{i-1} \cdot x_{i-1}^{-1} = x_i$ by induction, so α_n is onto, hence an automorphism with $\beta = \alpha_n^{-1}$ (Hopficity, or the direct check $\beta\alpha_n(x_i) = \beta(x_i)\beta(x_{i-1}) = x_i$). By induction $\beta(x_i)$ is a reduced word of length i with first letter x_i and remaining letters in $\{x_1, \ldots, x_{i-1}\}^{\pm}$: the tail $\beta(x_{i-1})^{-1}$ is reduced with letters below x_i, and its first letter is not x_i^{-1}, so no cancellation occurs. (Concretely $\beta(x_2) = x_2 x_1^{-1}$, $\beta(x_3) = x_3 x_1 x_2^{-1}$, $\beta(x_4) = x_4 x_2 x_1^{-1} x_3^{-1}$, …) Hence $\|\alpha_n^{-1}\| = \max_i |\beta(x_i)| = n$, while $\|\alpha_n\| = 2$ by definition.

(b) Write $\alpha := \alpha_n$; it preserves each $F_s := \langle x_1, \ldots, x_s \rangle$.

Step 0. For $s \geq 2$ consider the twisted equation

$$(E_s)\colon \qquad u = x_s \cdot \alpha(u) \cdot x_s^{-1}, \qquad u \in F_s.$$

Abelianizing (conjugation dies), with $\alpha_{\mathrm{ab}}(v)_j = v_j + v_{j+1}$ $(j < s)$ and $\alpha_{\mathrm{ab}}(v)_s = v_s$, the equation forces $v_2 = \cdots = v_s = 0$; in particular any solution has x_s-exponent sum 0.

Step 1 ((E_s) has only the trivial solution, $s \geq 2$). Let $u \neq 1$ solve (E_s); write $u = a_0\, x_s^{\eta_1}\, a_1 \cdots x_s^{\eta_r}\, a_r$ reduced, $a_j \in F_{s-1}$, $\eta_j = \pm 1$, $\sum \eta_j = 0$ by Step 0. Substituting images $(\alpha(x_s)^{+1} = x_s x_{s-1}$, $\alpha(x_s)^{-1} = x_{s-1}^{-1} x_s^{-1}$,

$\alpha(F_{s-1}) \subseteq F_{s-1}$), the right-hand side is a word with $r+2$ letters $x_s^{\pm 1}$:

$$x_s \cdot \alpha(a_0) \cdot T(\eta_1) \cdot \alpha(a_1) \cdots T(\eta_r) \cdot \alpha(a_r) \cdot x_s^{-1}, \qquad T(+1) = x_s x_{s-1}, \quad T(-1) = x_{s-1}^{-1} x_s^{-1}.$$

In free reduction, x_s-letters cancel only in nested consecutive pairs, and a consecutive opposite-sign pair cancels iff the x_s-free word between the two letters reduces to 1. The candidate sites: an inner pair $(\eta_j, \eta_{j+1}) = (+,-)$ needs $x_{s-1}\,\alpha(a_j)\,x_{s-1}^{-1} = 1$, i.e. $a_j = 1$, i.e. u contains $x_s x_s^{-1}$ — impossible; an inner $(-,+)$ pair needs $\alpha(a_j) = 1$ — impossible likewise; the front pair (initial x_s against $\eta_1 = -1$) needs $a_0 = \alpha^{-1}(x_{s-1})$ — possible; the back pair ($\eta_r = +1$ against the final x_s^{-1}) needs $a_r = \alpha^{-1}(x_{s-1}^{-1})$ — possible. The reduced right side must equal u, with r letters $x_s^{\pm 1}$, while the raw right side has $r+2$; each cancellation event removes exactly two, so exactly one event occurs, at one of the sites listed (the leading x_s and trailing x_s^{-1} become adjacent only at $r = 0$, where the outer pair must cancel and $u = 1$). If the front pair cancels, the surviving x_s-signs read $(\eta_2, \dots, \eta_r, -1)$ and must equal $(\eta_1, \dots, \eta_r)$; chasing equalities gives $\eta_j = -1$ for all j, so $\sum \eta = -r \neq 0$, contradicting Step 0. The back case symmetrically forces all $\eta_j = +1$. Hence no $u \neq 1$ exists.

Step 2 (top-generator peeling). Let $w \in \mathrm{Fix}(\alpha) \setminus \{1\}$ and let t be the largest index occurring in w; suppose $t \geq 3$. Write $w = u_0\, x_t^{\varepsilon_1}\, u_1 \cdots x_t^{\varepsilon_k}\, u_k$ reduced, $u_j \in F_{t-1}$, $k \geq 1$. The raw $\alpha(w)$ has exactly k letters $x_t^{\pm 1}$, and cancellation sites are the inner ones only, each requiring some $u_j = 1$ at an opposite-sign junction — excluded by reducedness. So $\alpha(w) = w$ matches syllable by syllable:

$$u_0 = \alpha(u_0) \cdot L(\varepsilon_1); \qquad u_j = R(\varepsilon_j) \cdot \alpha(u_j) \cdot L(\varepsilon_{j+1})\ (0 < j < k); \qquad u_k = R(\varepsilon_k) \cdot \alpha(u_k),$$

where $R(+1) = x_{t-1}$, $R(-1) = 1$, $L(-1) = x_{t-1}^{-1}$, $L(+1) = 1$. Apply the x_{t-1}-exponent homomorphism ϕ (which satisfies $\phi \circ \alpha = \phi$ on F_{t-1}): the u_0-equation forces $\varepsilon_1 = +1$, the u_k-equation forces $\varepsilon_k = -1$, and the middle ones force strict sign alternation; in particular $k \geq 2$ and $(\varepsilon_1, \varepsilon_2) = (+,-)$, whose equation is exactly (E_{t-1}) for u_1 with $t - 1 \geq 2$: by Step 1, $u_1 = 1$, making w unreduced — contradiction. Hence $t \leq 2$, i.e. $\mathrm{Fix}(\alpha_n) = \mathrm{Fix}(\alpha_2) \leq F_2$.

Step 3 (the F_2 case). The same peeling with $t = 2$, $F_1 = \langle x_1 \rangle$, $\alpha|_{F_1} = \mathrm{id}$: now (E_1) reads $u = x_1 u x_1^{-1}$, which holds for all $u \in F_1$, and the sign forcing as before yields fixed words exactly of the form

$$x_1^{p_0}\,(x_2 x_1^{q_1} x_2^{-1})\,x_1^{p_1}\,(x_2 x_1^{q_2} x_2^{-1}) \cdots x_1^{p_m}$$

(interior exponents nonzero) — all in $\langle x_1,\ x_2 x_1 x_2^{-1} \rangle$, since $x_2 x_1^q x_2^{-1} = (x_2 x_1 x_2^{-1})^q$. Conversely both generators are fixed: $\alpha(x_2 x_1 x_2^{-1}) = x_2 x_1 \cdot x_1 \cdot x_1^{-1} x_2^{-1} = x_2 x_1 x_2^{-1}$. The subgroup is nonabelian ($x_1 \cdot x_2 x_1 x_2^{-1} \neq x_2 x_1 x_2^{-1} \cdot x_1$ as reduced words), so it is free of rank exactly 2 and its two generators form a basis (a generating set of size equal to the rank is a basis, by Hopficity [11, Cor. 2.13.1]); by the normal form just displayed its elements of length at most 2 are powers of x_1, so a basis within the ball of radius 2 would generate an abelian subgroup, whence $M(\mathrm{Fix}(\alpha_n)) = 3$, attained by the displayed basis. □

4. The per-rank reading

4.1. **The bridge.** The positive half of the problem rests on the following theorem, published with two independent proofs.

Theorem 4.1 (Bogopolski–Maslakova [2, Thm. 1.1]; Feighn–Handel [6, Prop. 9.10])**.** *For every n there is a deterministic algorithm that, on every input n-tuple of reduced words representing an automorphism $\alpha \in \mathrm{Aut}\, F_n$, halts and outputs a finite sequence of words freely generating* $\mathrm{Fix}(\alpha)$.

This is exactly the input/output specification stated in [2] ("The input of this algorithm is the sequence of words $\phi(x_1), \dots, \phi(x_n) \dots$; the output are words in $X^{\pm}$ which freely generate $\mathrm{Fix}(\phi)$"); Proposition 9.10 of [6] is the same contract, by an independent argument (the survey restatement [13, Thm. 5.1] confirms the output form). Nothing is promised on non-automorphism inputs, and nothing will be used: our procedures filter first. The redundancy is deliberate — an algorithm published in 2003 [12] had a proof that "was not complete" by the later account of [2], and the problem stayed open until 2016 — and every use below may be instantiated by either source alone, so a refutation of Theorem 1.4 would refute both published theorems simultaneously.

Proof of Theorem 1.4. Fix n and an algorithm A as in Theorem 4.1.

A recursive bound exists. For $m \in \mathbb{N}$ the set $T(n, m)$ of n-tuples of reduced words of length at most m is finite and enumerable. Whether a tuple $\vec{w}$ defines an automorphism is decidable: $\varphi_{\vec{w}}$ is surjective iff $\langle w_1, \dots, w_n \rangle = F_n$, which the folded Stallings graph of $\{w_1, \dots, w_n\}$ decides — fold the wedge of circuits

spelling the w_i and test whether each basis letter x_j labels a closed path at the base vertex, membership in the subgroup represented by a folded graph being decidable by tracing [16, Alg. 5.4], [8, Prop. 3.8, Thm. 5.1] (see also the survey [3]) — and a surjective endomorphism of F_n is an automorphism, F_n being Hopfian [11, Thm. 2.13]. Now compute $f_n(m)$: enumerate $T(n,m)$, filter the automorphism tuples, run A on each — it halts, by Theorem 4.1, with a basis of the fixed subgroup — and output the maximum length of a word appearing in any of the outputs (0 if no tuple is accepted, which happens exactly at $m=0$: for $m\geq 1$ the identity tuple is accepted). This is a single-process finite search around halting calls, with no step-counting, simulation, or dovetailing, so f_n is total recursive. For $\alpha\in\operatorname{Aut}F_n$ the tuple $\vec{w}_\alpha=(\alpha(x_1),\dots,\alpha(x_n))$ lies in $T(n,\|\alpha\|)$ and is accepted, so $\operatorname{Fix}(\alpha)$ has a basis — the one A outputs — with all elements of length at most $f_n(\|\alpha\|)$. This proves the per-rank reading.

The optimal bound is computable. Upgrade the maximand from "longest output word" to $M(\operatorname{Fix}(\varphi_{\vec{w}}))$, which is computable from the output basis $Y=\{y_1,\dots,y_k\}$ (reduce the output words first; a freely generating sequence has distinct nontrivial entries). If $k=0$, $M=0$. Otherwise, for $L=1,2,\dots$: compute $S_L:=\{h\in H:|h|\leq L\}$, where $H:=\langle Y\rangle$ — finitely many candidates, membership decided by the folded graph of Y — and for every k-element subset $T\subseteq S_L$ decide whether $\langle T\rangle=H$: since $T\subseteq H$, this holds iff every $y_j\in\langle T\rangle$, decidable by the folded graph of T. Output the least L at which some T succeeds. The search terminates: at $L=\max_j|y_j|$ the subset $T=Y$ succeeds. Its output is $M(H)$: every basis of H has exactly k elements [10, Prop. I.3.5], so a basis realizing $M(H)$ is a candidate T at level $M(H)$, giving output $\leq M(H)$; conversely a successful T is a k-element generating set of the free group H of rank k, hence a basis of H (Hopficity again [11, Cor. 2.13.1]), with all elements of length at most the output level. With this maximand the same enumeration computes $f_n(m)=\max\{M(\operatorname{Fix}(\alpha)):\|\alpha\|\leq m\}=f^\circ(n,m)$ exactly, the accepted tuples being norm-preservingly in bijection with the automorphisms of norm at most m. □

Proof of Theorem 1.5. (1) ⇒ (2): let g be recursive and valid for $\operatorname{Aut}F_n$. On input the tuple $\vec{w}$ of an automorphism α: compute $L:=g(\max_i|w_i|)=g(\|\alpha\|)$, then the finite set $S:=\{u \text{ reduced}:|u|\leq L,\ \alpha(u)=u\}$ (fixedness is the substitution test of Section 1). Then $\langle S\rangle=\operatorname{Fix}(\alpha)$: the inclusion $\leq$ holds since $S\subseteq\operatorname{Fix}(\alpha)$ and $\operatorname{Fix}(\alpha)$ is a subgroup; conversely, by hypothesis *some* basis $\{y_1,\dots,y_k\}$ of $\operatorname{Fix}(\alpha)$ has all $|y_i|\leq L$, and its elements are fixed words of length at most L, hence lie in S, so $\operatorname{Fix}(\alpha)=\langle y_1,\dots,y_k\rangle\leq\langle S\rangle$. (The hypothesis supplies only some short basis, not that all short fixed words form one; a short basis inside S is exactly enough.) Finally fold S: the folded graph represents $\langle S\rangle=\operatorname{Fix}(\alpha)$, and a maximal tree yields a free basis of it [16, Alg. 5.4]. Every step terminates. (2) ⇒ (1): the proof of Theorem 1.4 invoked Theorem 4.1 only through the stated contract — a total algorithm producing a basis of the fixed subgroup from the tuple — which is statement (2). □

Remark 4.2 (scope of the citations). The quoted theorems assert a *total, correct* algorithm on the set of automorphism inputs; if the cited algorithms were partial on some automorphism, the proof above would break. That risk is exactly the correctness of the two published theorems, doubly covered. On "efficient": Problem A of [2] asks for an algorithm with a recursive step-count bound, no two-process tricks, no diagonalization; our derived algorithms are single-process finite searches around calls to A and inherit that character from A. We claim nothing finer — the mathematical content used is totality plus correctness.

Remark 4.3 (uniformity in n). Theorem 1.4 is per-rank. The stronger statement — that $(n,m)\mapsto f^\circ(n,m)$ is computed by a single algorithm of two arguments — follows verbatim from the same proof provided the Fix-basis algorithm is a single algorithm for all n, the tuple determining n. That is the natural reading of [6] ("There is an algorithm that, given $\Phi\in\operatorname{Aut}(F_n)$, computes $\operatorname{Fix}(\Phi)$"), of the survey restatement [13, Thm. 5.1], and of the input specification of [2]; nothing in either proof chooses anything non-uniformly in n. We record the uniform version as conditional only on reading the cited theorems as stated for a single algorithm; under the weakest reading (for each n an algorithm A_n), Theorem 1.4 survives unchanged.

Remark 4.4 (no explicit upper bound is available). Theorem 1.4 produces the optimal $f^\circ(n,\cdot)$, but through calls to algorithms with no published complexity bound, and that is all we claim. No explicit bound (elementary, primitive recursive, or otherwise) can be extracted from the published literature, because none is published for the underlying algorithms; the only printed complexity commentary we found describes them as "making strong use of the train-track technology, and providing quite complicated and high complexity algorithms" [14]. Any future explicit step bound $T(n,m)$ for a Fix-basis algorithm would immediately explicitize Theorem 1.4: $f^\circ(n,m)\leq T(n,m)$, an algorithm being unable to output a word longer than its

step count. The gap between the linear lower bound below and "recursive" is enormous and entirely open, even for $n = 2$.

4.2. **Lower bounds.**

Theorem 4.5 (inner automorphisms force a linear lower bound)**.** *Let $n \geq 2$ and $\ell \geq 1$. Put $g := x_1^{\ell-1}x_2 \in F_n$ (so $g = x_2$ when $\ell = 1$; $|g| = \ell$) and let $\gamma_g \in \operatorname{Aut} F_n$ be the inner automorphism $u \mapsto gug^{-1}$. Then $\|\gamma_g\| = 2\ell + 1$, and $\operatorname{Fix}(\gamma_g) = \langle g \rangle$, whose only bases are $\{g\}$ and $\{g^{-1}\}$, of length exactly ℓ. Consequently every f valid for F_n in the sense of Problem 14.23 — for any single $n \geq 2$, hence also any uniform f — satisfies*

$$f(2\ell + 1) \geq \ell \quad (\ell \geq 1), \qquad \textit{i.e.} \quad f(m) \geq (m-1)/2 \text{ for odd } m \geq 3,$$

and $f^\circ(n, m) \geq \lfloor (m-1)/2 \rfloor$ for all $m \geq 3$ (monotonicity, applied to the largest odd $m' \leq m$). In particular no constant, and no sublinear, f works under either reading.

Proof. Norms: $\gamma_g(x_1) = x_1^{\ell-1}x_2x_1x_2^{-1}x_1^{-(\ell-1)}$ is reduced as written (no junction cancels; for $\ell = 1$ read x_1^0 as empty), of length $2\ell + 1$; $\gamma_g(x_2) = x_1^{\ell-1}x_2x_1^{-(\ell-1)}$, length $2\ell - 1$; and for $i \geq 3$, $\gamma_g(x_i) = x_1^{\ell-1}x_2x_ix_2^{-1}x_1^{-(\ell-1)}$, reduced of length $2\ell + 1$. So $\|\gamma_g\| = 2\ell + 1$. Fixed subgroup: $\operatorname{Fix}(\gamma_g) = \{u : ug = gu\} = C(g) = \langle c \rangle$ with $g = c^s$ (Lemma 2.1); abelianizing, the x_2-coordinate of the image $(\ell - 1, 1, 0, \ldots, 0)$ of g reads $s \cdot (\text{integer}) = 1$, so $s = \pm 1$ and $\operatorname{Fix}(\gamma_g) = \langle g \rangle$, whose bases are $\{g^{\pm 1}\}$, of length ℓ. Given a valid f, the promised basis of $\operatorname{Fix}(\gamma_g)$ at norm $2\ell + 1$ forces $f(2\ell + 1) \geq \ell$. □

Proposition 4.6 (the value at $(n, m) = (2, 2)$)**.** *Let $n = 2$ with basis $\{a, b\}$, and let $\alpha \in \operatorname{Aut} F_2$ be defined by $\alpha(a) = a$, $\alpha(b) = ab$ (inverse: $a \mapsto a$, $b \mapsto a^{-1}b$). Then $\|\alpha\| = 2$ and every basis of $\operatorname{Fix}(\alpha)$ contains an element of length at least 3. Hence any valid f for $n = 2$ satisfies $f(2) \geq 3 > 2$, and $f^\circ(2, 2) \geq 3$; in particular "$f(m) = m$" is not a valid bound, already at $m = 2$.*

Proof. $K := \langle a,\ b^{-1}ab \rangle \leq \operatorname{Fix}(\alpha)$: indeed $\alpha(b^{-1}ab) = (ab)^{-1} \cdot a \cdot (ab) = b^{-1}ab$. K is nonabelian: $ab^{-1}ab$ and $b^{-1}aba$ are distinct reduced words, so the two generators do not commute. A direct check of the 16 nontrivial reduced words of length at most 2 shows the fixed ones are exactly $a^{\pm 1}, a^{\pm 2}$ (each is a one-line substitution, e.g. $\alpha(a^{-1}b) = a^{-1} \cdot ab = b \neq a^{-1}b$). A basis of $\operatorname{Fix}(\alpha)$ with all elements of length at most 2 would therefore lie in $\langle a \rangle$, making $\operatorname{Fix}(\alpha)$ abelian — contradicting $K \leq \operatorname{Fix}(\alpha)$. So $M(\operatorname{Fix}(\alpha)) \geq 3$ while $\|\alpha\| = 2$. □

In fact $\operatorname{Fix}(\alpha) = \langle a,\ b^{-1}ab \rangle$ exactly — the classical example of Dyer and Scott [5] (see [18, Thm. 2.1]); the proof above deliberately avoids the exact equality. The true value of $f^\circ(2, 2)$ is 4, attained by an automorphism of a different, non-elementary kind (Section 5).

5. Computations: certified small values

The theorems above bound f° from below by explicit families; this section reports what complete machine computation gives for small n and m. All computations were carried out twice, in independent toolchains: in GAP 4.15.1 [17] with the package FGA 1.5.0 [15], and in word-arithmetic and folding routines written from scratch for this project (see the final section). We performed complete sweeps of all automorphisms of F_2 of norm at most 4 (1,032 automorphisms), of F_3 of norm at most 2 (8,112), and of F_4 of norm at most 2 (1,886,592), quotienting by the norm-preserving symmetries $\alpha \mapsto \sigma\alpha^{\pm 1}\sigma^{-1}$ with σ a signed permutation — these carry bases of fixed subgroups to bases preserving lengths, since $\operatorname{Fix}(\sigma\alpha\sigma^{-1}) = \sigma(\operatorname{Fix}\alpha)$ and $\operatorname{Fix}(\alpha^{-1}) = \operatorname{Fix}(\alpha)$ — leaving 122, 180 and 5,010 classes respectively.

A value is reported as *certified* only when backed by a finite, machine-checkable certificate of one of two kinds. The first:

Lemma 5.1 (complete-ball certificate)**.** *Let $\alpha \in \operatorname{Aut} F_n$, let $B \geq 1$, let $S = \operatorname{Fix}(\alpha) \cap \{w : |w| \leq B\}$ (computable by finite enumeration), and suppose some $e \in \operatorname{Fix}(\alpha)$ satisfies $e \notin \langle S \rangle$ (decidable by folding). Then every basis of $\operatorname{Fix}(\alpha)$ contains an element of length greater than B, i.e. $M(\operatorname{Fix}(\alpha)) > B$ — regardless of whether $\operatorname{Fix}(\alpha)$ is known.*

Proof. A basis Y with all elements of length at most B consists of fixed words of length at most B, so $Y \subseteq S$ and $\operatorname{Fix}(\alpha) = \langle Y \rangle \leq \langle S \rangle \not\ni e \in \operatorname{Fix}(\alpha)$ — a contradiction. □

The second kind is the inner-power mechanism of Lemma 2.9: if $\alpha^k = \gamma_g$ is verified on every generator, for an explicit $g \neq 1$, then $\mathrm{Fix}(\alpha) = \langle \hat{g} \rangle$ *exactly*, independent of any enumeration horizon; root-freeness of a cyclically reduced g is itself a finite check ($g = u^d$ with $d \geq 2$ forces g to be the literal d-fold repetition of its length-$|g|/d$ prefix). Three instances carry the headline values.

Proposition 5.2. (i) *In $F_2 = F(a,b)$, let $\alpha\colon a \mapsto b^{-1},\ b \mapsto ab$. Then $\|\alpha\| = 2$, the abelianization of α has order 6 in $\mathrm{GL}_2(\mathbb{Z})$, and $\alpha^6 = \gamma_g$ with $g = [b,a] = b^{-1}a^{-1}ba$, root-free. Hence $\mathrm{Fix}(\alpha) = \langle [b,a] \rangle$, whose only bases are $\{[b,a]^{\pm 1}\}$, so $M(\mathrm{Fix}(\alpha)) = 4$ and $f^\circ(2,2) \geq 4$.*

(ii) *In F_2, let $\alpha\colon a \mapsto b^{-1}a^{-1}b^{-2},\ b \mapsto b^2a$. Then $\|\alpha\| = 4$ and $\alpha^6 = \gamma_g$ with*

$$g = b^2ab^2abab^{-2}a^{-1}b^{-2}a^{-1}b^{-1}a^{-1}$$

of length 16, root-free. Hence $\mathrm{Fix}(\alpha) = \langle g \rangle$, $M(\mathrm{Fix}(\alpha)) = 16$, and $f^\circ(2,4) \geq 16$.

(iii) *In $F_4 = F(a,b,c,d)$, let $\alpha\colon a \mapsto b^{-1},\ b \mapsto c^{-1}b,\ c \mapsto d^{-1}b,\ d \mapsto ac$. Then $\|\alpha\| = 2$, the abelianization of α has order 12 in $\mathrm{GL}_4(\mathbb{Z})$, and $\alpha^{12} = \gamma_g$ with*

$$g = b^{-1}cd^{-1}bc^{-1}a^{-1}c^{-1}d^2acacb^{-1}d^{-2}a^{-1}c^{-1}a^{-1}dba$$

of length 22, root-free. Hence $\mathrm{Fix}(\alpha) = \langle g \rangle$, $M(\mathrm{Fix}(\alpha)) = 22$, and $f^\circ(4,2) \geq 22$.

Proof. In each case the automorphism property is a finite folding computation, the identity $\alpha^k = \gamma_g$ is a finite computation on the generators, and root-freeness of the cyclically reduced g is the finite periodicity check above; all were performed independently in both toolchains. Lemma 2.9 then gives $\mathrm{Fix}(\alpha) = \langle g \rangle$ (here $\hat{g} = g$), whose bases are $\{g^{\pm 1}\}$, whence $M(\mathrm{Fix}(\alpha)) = |g|$. □

Proposition 5.3. *In $F_3 = F(a,b,c)$, let $\alpha\colon a \mapsto b^{-1},\ b \mapsto a,\ c \mapsto a^{-1}c$, of norm 2. Complete enumeration shows that α fixes no nontrivial word of length at most 5, while the length-6 word $w = c^{-1}b^{-1}aba^{-1}c$ is fixed:*

$$\alpha(w) = (c^{-1}a)(a^{-1})(b^{-1})(a)(b)(a^{-1}c) = c^{-1}b^{-1}aba^{-1}c = w.$$

By Lemma 5.1 (with $B = 5$, $S = \{1\}$, $e = w$), every basis of $\mathrm{Fix}(\alpha)$ contains an element of length at least 6; hence $f^\circ(3,2) \geq 6$.

The certified values, and the status of the exact evaluations, are as follows. (The automorphism of Proposition 5.2(i) is the standard order-six mapping class of the once-punctured torus; its fixed subgroup is generated by the boundary word.)

value	certified content	status of the exact value
$f^\circ(n,1) = 1$, all n	proved (Proposition 2.10)	exact (independently re-derived by the sweep)
$f^\circ(2,2) = 4$	≥ 4: Proposition 5.2(i), doubly (also a ball certificate)	exact on all norm-2 classes but one; see below
$f^\circ(2,3) = 4$	≥ 4	exact certificates for 27 of 54 classes; rest by stable enumeration
$f^\circ(2,4) = 16$	≥ 16: Proposition 5.2(ii)	$= 16$ modulo 76 classes decided by stable enumeration only
$f^\circ(3,2) \geq 6$	Proposition 5.3	the value 6 is the stable-sweep value; not certified
$f^\circ(4,2) \geq 22$	Proposition 5.2(iii)	the value of $f^\circ(4,2)$ is not settled

The one norm-2 class of F_2 without an exact certificate is that of the Fibonacci automorphism $a \mapsto ab$, $b \mapsto a$: the computations found no nontrivial fixed element (none of length at most 14), and homology confines any fixed element to the commutator subgroup (the abelianization M satisfies $\det(M - I) = -1 \neq 0$), but no proof of $\mathrm{Fix} = 1$ is included here; granting it, $f^\circ(2,2) = 4$ is certificate-exact. "Stable enumeration" means that the list of fixed words of length at most L showed no change over a widening window of L; the next remark is the reason we treat such evidence as evidence only.

Remark 5.4 (hidden roots). The automorphism of Proposition 5.2(ii) has no nontrivial fixed word of length at most 10, yet its fixed subgroup is infinite cyclic on a word of length 16; likewise (iii) has no fixed word of length at most 6 while $|g| = 22$. Pure enumeration to any fixed depth would have labeled both classes trivial; this is why the table separates certificates from stabilization so strictly.

The certified sequence $f^\circ(n,2) = 4,\ \geq 6,\ \geq 22$ at $n = 2,3,4$ dominates the floor $f^\circ(n,2) \geq n$ of Theorem 1.2; the mechanism visible in Proposition 5.2 — norm-2 automorphisms whose abelianization has finite order (the new finite orders 8, 10, 12 in $\mathrm{GL}_4(\mathbb{Z})$ enable the jump at $n = 4$) with a long inner-power conjugator — is what suggested the construction of ζ_r. In the other variable, $f^\circ(2,m) = 1,4,4,16$ for $m = 1,\dots,4$ — already $\geq m^2$ at $m = 2,4$ — so the per-rank recursive bound sits far above the linear floor of Theorem 4.5 even at $n = 2$. The computations also corroborate the families at every point they can touch: for ζ_r ($r \leq 12$), the automorphism property, both norms, the identity $\zeta^{2r} = \gamma_Z$ on every generator, the primitivity of Z, and exhaustive enumerations finding no fixed words other than powers of Z within range; for β_n ($n \leq 6$), the same, including the complete scan of all 2,929,686 nontrivial reduced words of length at most 9 in F_3, whose fixed members are exactly $C^{\pm 1}, C^{\pm 2}, C^{\pm 3}$; for the shift-append family ($n \leq 6$), that every fixed word within range lies in $\langle x_1, x_2 x_1 x_2^{-1} \rangle$, with per-length fixed counts independent of n, and the inverse-norm pattern $\|\alpha_n^{-1}\| = n$ for $n \leq 10$.

6. Open questions

(1) *Exact small values.* $f^\circ(3,2)$ is certified only ≥ 6; the stable-sweep value is 6. Similarly $f^\circ(4,2) \geq 22$ (value unsettled) and $f^\circ(2,4) = 16$ modulo classes decided by stable enumeration. The one remaining norm-2 class of F_2 — the Fibonacci automorphism $a \mapsto ab$, $b \mapsto a$ — needs a proof that its fixed subgroup is trivial (its fixed elements lie in the commutator subgroup, and none exists of length at most 14; the triviality is presumably known to experts, but we found no printed proof) to finish $f^\circ(2,2) = 4$ entirely by certificates.

(2) *Upper bounds on $f^\circ(n,2)$.* None are known. The certified growth $4, \geq 6, \geq 22$ at $n = 2,3,4$ against the proved floor $f^\circ(n,2) \geq n$, and the fact that the finite orders in $\mathrm{GL}_n(\mathbb{Z})$ grow unboundedly with n, suggest super-linear growth; the experiments record a candidate norm-2 family whose fixed generators appear to grow like $10n/3$, with a clean structure depending on n mod 4, but its inner-power identity is unproved — proving it, for instance by a graph realization in the style of Remark 2.12, is a natural next step.

(3) *Growth of the optimal per-rank bounds.* $f_n = f^\circ(n,\cdot)$ is recursive (Theorem 1.4) with certified values $1,4,4,16$ at $n = 2$, $m = 1,\dots,4$, but no explicit (even primitive recursive) upper bound is extractable from the published literature (Remark 4.4). Any published step-count bound for a Fix-basis algorithm would immediately explicitize Theorem 1.4.

(4) *Uniformity of the algorithm across n.* Remark 4.3's conditional statement — that $(n,m) \mapsto f^\circ(n,m)$ is recursive as a single two-argument function — holds under the natural single-algorithm reading of [2, 6]; pinning that reading down from the sources is a mild open bookkeeping point (the per-rank Theorem 1.4 needs none of it).

(5) *Problem 14.24.* The companion problem [9, Problem 14.24] (same author, same norm) asks for a recursive $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ bounding, for conjugate $\alpha, \beta \in \mathrm{Aut}\, F_n$, the norm of some conjugator in terms of $(\|\alpha\|, \|\beta\|)$. By the same enumeration bridge as in Section 4, its per-rank reading is equivalent to the decidability of the conjugacy problem in $\mathrm{Aut}\, F_n$, which is, to our knowledge, open in general — so 14.24 remains open under every reading, unlike 14.23. Whether the length-versus-norm mechanisms of Sections 2 and 3 bear on its uniform, two-norm version is untested.

Provenance and verification

This note was produced by an automated research pipeline built on a large language model, `claude-fable-5`, and its claims were verified before assembly: each of the three claim packages — the per-rank theory, the norm-two family, the norm-three material — passed two independent adversarial verification passes (every lemma re-derived, every computation independently reprogrammed from the stated data), a citation audit checking every reference quoted above against an independently fetched source, and a novelty audit. Every computational claim was reproduced on independent implementations: GAP 4.15.1 [17] with FGA 1.5.0 [15] on one side, and from-scratch word-arithmetic, folding and enumeration engines (sharing no code, some written by the verifying agents from the statements alone) on the other; every load-bearing row of the table in Section 5 carries a finite machine-checkable certificate (Lemmas 5.1 and 2.9), re-verified in an independent GAP battery with verifying conjugator extraction (a returned g is checked on every generator against γ_g, so a false positive is impossible); $\zeta^{2r} = \gamma_Z$ was re-derived symbolically up to $r = 60$, and the twisted-equation

analysis of Proposition 3.5 was brute-forced over 9.5 million words. Complete machine records — scripts, transcripts, sweep tables, verifier and audit reports — are preserved in the repository accompanying this note and are available from the authors.

To our knowledge — a statement about the searches performed, not an absolute claim — the following are new here: the two-reading analysis of Problem 14.23 and both answers; the enumeration bridge of Theorem 1.4 with the optimality refinement $f_n = f^\circ(n, \cdot)$; the equivalence of Theorem 1.5 with its historical Remark 1.6; the quantitative floor (Theorem 4.5, Propositions 2.10, 4.6 and 2.11); the family ζ_r with the power identity $\zeta^{2r} = \gamma_Z$ and the sharp-threshold refutation $F(2) = \infty$; the exact equality $\mathrm{Fix}(\beta_n) = \langle C \rangle$ for the periodic-braid family (the family itself is classical — Artin — via [7]); Proposition 3.5; and the certified values of Section 5, including $f^\circ(2, 2) = 4$. Everything else is classical and cited where used; for the general theory of fixed subgroups see [18, 4] — the closest printed relatives our searches found are rank- and inertness-shaped results in that tradition, never length bounds against the norm. Three caveats of scope: the authors are aware of independent unpublished work on this problem, not compared with the present text at the time of writing; MathSciNet was not consulted in the searches performed (zbMATH was, through its public interface); and the full texts of [10] and [4] were not examined — nothing load-bearing rests on them (the centralizer lemma is proved in the text, the basis facts carry parallel citations) — while the published version of [2] was verified through its review and its arXiv text rather than the journal typesetting.

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