A COUNTEREXAMPLE TO KOUROVKA NOTEBOOK PROBLEM 14.67

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Abstract. We describe a finite group of order 660 602 880 with an involution having a largest centralizer among all nonidentity elements, and a normal subgroup of nilpotency class two that this involution does not centralize. The construction gives a negative answer to problem 14.67. The proof is self-contained and does not depend on a computer enumeration.

1. Introduction

Problem 14.67 of the Kourovka Notebook [1, p. 79], posed by I. T. Mukhamet'yanov and A. N. Fomin, asks the following question.

Question. *Suppose that a is a non-trivial element of a finite group G such that $|C_G(a)| \geq |C_G(x)|$ for every non-trivial element $x \in G$, and let H be a nilpotent subgroup of G which is normalized by $C_G(a)$. Is it true that $H \leq C_G(a)$?*

The Notebook records an affirmative answer when H is abelian, and also when H is a p'-group for some prime $p \in \pi(Z(C_G(a)))$. Here $C_G(x)$ denotes the centralizer of x, $Z(X)$ the center of a group X, and $\pi(X)$ the set of primes dividing $|X|$. We give a counterexample with H normal and of nilpotency class two.

Theorem 1. *There is a group G of order* 660 602 880 *with a nonidentity involution a and a normal subgroup H of order* 16 384 *and nilpotency class two such that*

$$|C_G(a)| = 41\,287\,680 \geq |C_G(g)| \quad (1 \neq g \in G), \qquad H \not\leq C_G(a).$$

In particular, the answer to problem 14.67 is negative, even when H is required to be normal in G.

2. Construction of the group

All vector spaces and exterior products in this note are over $\mathbb{F}_2$. Let $E = \mathbb{F}_2^4$, with basis e_1, e_2, e_3, e_4, and let $W = \Lambda^2 E$, of dimension six. Recall that in characteristic two $u \wedge v = v \wedge u$ and $u \wedge u = 0$.

On $P = E \times E \times W$ define

$$(x, y, z)(s, t, w) = (x + s, y + t, z + w + x \wedge t). \tag{1}$$

Bilinearity of $\wedge$ gives associativity: the third coordinate of either association of three factors contains the same three cross terms. The identity is $(0, 0, 0)$, and

$$(x, y, z)^{-1} = (x, y, z + x \wedge y).$$

Thus P is a group of order $2^{14} = 16\,384$.

Put $L = \mathrm{GL}(E)$. For $R \in L$, write $R_2 = \Lambda^2 R$ and define

$$\beta_R(x, y, z) = (Rx, Ry, R_2 z), \qquad \alpha(x, y, z) = (y, x, z + x \wedge y).$$

Naturality of the exterior product shows that β_R is an automorphism and that $\beta_R\beta_S = \beta_{RS}$. Expanding (1) shows that α is an automorphism as well. Indeed, for $u = (x, y, z)$ and $v = (s, t, w)$, the third coordinate of $\alpha(uv)$ simplifies to $z + w + x \wedge y + s \wedge t + s \wedge y$, exactly that of $\alpha(u)\alpha(v)$. Also $\alpha^2 = 1$, and $\alpha(e_1, 0, 0) = (0, e_1, 0)$, so $\alpha \neq 1$. Finally $\alpha\beta_R = \beta_R\alpha$.

Consequently $K = L \times C_2$ acts on P, with the generator of C_2 acting by α. Define

$$G = P \rtimes K, \qquad H = P, \qquad a = ((0, 0, 0), I, 1).$$

More explicitly, elements of G are triples (h, R, ε), where $h \in P$, $R \in L$, and $\varepsilon \in \{0,1\}$, with

$$(h, R, \varepsilon)(h', S, \delta) = (h\,\beta_R(\alpha^{\varepsilon}(h')), RS, \varepsilon + \delta \pmod 2).$$

We identify H with $\{(h, I, 0) : h \in P\}$ and W with its central copy $\{(0,0,z) : z \in W\}$ in P. Since

$$|L| = (16-1)(16-2)(16-4)(16-8) = 20\,160,$$

we have $|G| = 16\,384 \cdot 20\,160 \cdot 2 = 660\,602\,880$.

The normal subgroup. Using $[u, v] = u^{-1}v^{-1}uv$, direct multiplication gives

$$[(x, y, z), (s, t, w)] = (0, 0, x \wedge t + s \wedge y). \tag{2}$$

Thus $[P, P] \leq W \leq Z(P)$. Every basis bivector $e_i \wedge e_j$ is the commutator of $(e_i, 0, 0)$ and $(0, e_j, 0)$, so $[P, P] = W$. If $x \neq 0$, choose t with $x \wedge t \neq 0$ and put $s = 0$ in (2); the case $y \neq 0$ is analogous. Hence $Z(P) = W$. The subgroup $H = P$ therefore has nilpotency class exactly two. It is normal in G, so is normalized by $C_G(a)$. Nevertheless

$$a(e_1, 0, 0)a^{-1} = (0, e_1, 0) \neq (e_1, 0, 0),$$

where elements of P are viewed inside G. Thus $H \nleq C_G(a)$.

3. Comparison of centralizer orders

Since a is central in K, an element $(h, k) \in P \rtimes K$ commutes with a exactly when $\alpha(h) = h$. The fixed elements of α in P are precisely (x, x, z), with $x \in E$ and $z \in W$. There are $16 \cdot 64 = 1024$ such elements. It follows that

$$|C_G(a)| = 1024 \cdot 40\,320 = 41\,287\,680, \qquad |a^G| = 16.$$

By the class-size formula, it remains to show that every nonidentity element of G has at least sixteen conjugates. We consider three cases.

Case 1: nonzero elements of W. Both P and a centralize W, so a G-conjugacy class in W is exactly an L-orbit under the exterior-square action. There are two nonzero orbits, represented by

$$e_1 \wedge e_2 \quad \text{and} \quad e_1 \wedge e_2 + e_3 \wedge e_4.$$

For completeness, choose a nonzero coefficient of a bivector and relabel the basis so that it is the $e_1 \wedge e_2$ coefficient. Writing $U = \langle e_3, e_4 \rangle$, the bivector has the form

$$e_1 \wedge e_2 + e_1 \wedge u + e_2 \wedge v + w = (e_1 + v) \wedge (e_2 + u) + (w + u \wedge v),$$

with $u, v \in U$ and $w \in \Lambda^2 U$. The first two new vectors complement U, and the final summand is either zero or the unique nonzero element of $\Lambda^2 U$. This yields the two displayed forms. They cannot be equivalent: their associated alternating matrices have ranks two and four, respectively, and rank is invariant under change of basis.

Each two-dimensional subspace of E has exactly one nonzero bivector. Thus the first orbit has size

$$\frac{(16-1)(16-2)}{(4-1)(4-2)} = 35.$$

The other orbit has size $(2^6 - 1) - 35 = 28$. Both exceed sixteen.

Case 2: elements of $P \setminus W$. Let $h = (x, y, z)$ with $(x, y) \neq (0, 0)$. By (2), its P-conjugates are parametrized by the image of the linear map

$$E \oplus E \longrightarrow W, \qquad (s, t) \longmapsto x \wedge t + s \wedge y.$$

This image has dimension at least three: if $x \neq 0$ it contains $x \wedge E$, of dimension three, and if $y \neq 0$ it contains $E \wedge y$. Hence $|h^P| \geq 8$.

The diagonal L-orbit of (x, y) has at least fifteen elements: projection onto any nonzero coordinate maps that orbit onto all fifteen nonzero vectors of E. These quotient elements represent distinct cosets of W in P. In each such coset the corresponding L-conjugate of h has at least eight P-conjugates, all belonging to h^G. The cosets are disjoint, so

$$|h^G| \geq 15 \cdot 8 = 120 > 16.$$

Case 3: elements of $G \setminus P$. Write $g = (h, R, \varepsilon)$, where $(R, \varepsilon) \neq (I, 0)$. Conjugation by $h \in P$ acts trivially on both P/W and W. On $V = P/W \cong E \oplus E$, the map $p \mapsto gpg^{-1}$ therefore induces

$$T(x,y) = \begin{cases} (Rx, Ry), & \varepsilon = 0, \\ (Ry, Rx), & \varepsilon = 1, \end{cases}$$

and its action on W is R_2. Set $r_V = \operatorname{rank}(T - I)$ and $r_W = \operatorname{rank}(R_2 - I)$. The projection of $C_P(g)$ into P/W lies in $\operatorname{Fix}(T)$; its kernel is exactly $C_W(g) = \operatorname{Fix}(R_2)$. Consequently

$$|C_P(g)| \leq 2^{8-r_V} 2^{6-r_W}, \qquad |g^P| \geq 2^{r_V + r_W}. \tag{3}$$

This upper bound does not require every fixed quotient element to lift to an element of $C_P(g)$.

If $\varepsilon = 1$, a fixed pair satisfies $y = Rx$, so is determined by x. Thus $\dim \operatorname{Fix}(T) \leq 4$ and $r_V \geq 4$.

If $\varepsilon = 0$, then $R \neq I$ and $r_V = 2 \operatorname{rank}(R - I)$. This is at least four unless $\operatorname{rank}(R - I) = 1$. In that exceptional case write $R - I = uf$, where $u \neq 0$ and f is a nonzero linear functional. Invertibility forces $f(u) = 0$, since otherwise $R(u) = u + u = 0$. Choose a basis in which $u = e_1$, $f(e_2) = 1$, and $e_1, e_3, e_4 \in \ker f$. Then $R(e_2) = e_2 + e_1$, with the other three basis vectors fixed. Directly on the six basis bivectors,

$$\operatorname{im}(R_2 - I) = \langle e_1 \wedge e_3, e_1 \wedge e_4 \rangle.$$

Therefore $r_W = 2$ and $r_V + r_W = 2 + 2 = 4$ also in this case. Equation (3) now yields $|g^G| \geq |g^P| \geq 16$ for every $g \notin P$.

The three cases exhaust all nonidentity elements of G. Since $|a^G| = 16$, the identity $|g^G| = |G|/|C_G(g)|$ proves the required maximum-centralizer inequality. Together with the construction and the failure of centralization, this proves the theorem. □

4. Relation to known positive results

The example avoids both affirmative cases recorded in the Notebook. The subgroup $H = P$ is nonabelian, and

$$Z(C_G(a)) = \langle a \rangle \cong C_2.$$

Indeed, $C_G(a) = C_P(a) \rtimes K$. A central element projects into $Z(K) = \langle a \rangle$, because $Z(\mathrm{GL}_4(2)) = 1$. Its P-component must be fixed by every element of L. An L-fixed element (x, x, z) of $C_P(a)$ has $x = 0$; Case 1 shows that $z = 0$ as well. Conversely, a is central in its own centralizer. Thus the only relevant prime in the second affirmative case is 2, whereas H is a nontrivial 2-group.

A related result of Yadav [2, Lemma 2.1] gives a strict increase from $|C_G(b)|$ to $|C_G([b, n])|$ for a nontrivial commutator $[b, n]$, provided that $N \trianglelefteq G$, $b \notin Z(G)$, and the *set* $\{[b, n] : n \in N\}$ is invariant under conjugation by N. This additional condition fails here. Direct calculation gives

$$[a, (x, y, z)] = (x + y, x + y, 0), \qquad \{[a, h] : h \in P\} = \{(u, u, 0) : u \in E\}.$$

For example, conjugating $(e_1, e_1, 0)$ by $(0, e_2, 0)$ gives $(e_1, e_1, e_1 \wedge e_2)$, which is not in this set. In particular, nilpotency of P, even of class two, does not supply the extra normality assumption in that lemma.

No assertion is made that the order of this counterexample is minimal. The proof does not rely on computer calculations.

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References

[1] E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved Problems in Group Theory. The Kourovka Notebook*, 21st ed., Sobolev Institute of Mathematics, Novosibirsk, 2026; arXiv:1401.0300v46 (1 September 2026), Problem 14.67, p. 79.

[2] M. K. Yadav, *On subgroups generated by small classes in finite groups*, Comm. Algebra **41** (2013), no. 9, 3350–3354. doi:10.1080/00927872.2012.686126. Preprint: arXiv:0905.2674v3.

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