A FINITE COUNTEREXAMPLE TO KOUROVKA NOTEBOOK 16.53

Abstract. We construct finite groups $G = \langle A, B\rangle$ with coprime subgroup orders, $d(A) = d(B) = 2$, and $d(G) = 4$. The construction uses coordinate functions in a finite power of $\mathrm{SL}_2(\mathbb{F}_{11})/\{I, -I\}$. Exact counts in the group of order 660 give more coordinate homomorphisms than a three-generated group can admit. Two reproducible Python checks accompany this draft; the additional GAP reproduction remains pending.

1. Problem and construction

Kourovka Notebook Problem 16.53 asks:

> Let $d(G)$ denote the smallest cardinality of a generating set of the group G. Suppose that $G = \langle A, B\rangle$, where A and B are two d-generated finite groups of coprime orders. Is it true that $d(G) \le d + 1$?

The archived statement and its bibliographic context are preserved in `source/task.json`; the extracted generator brackets are normalized here.

Theorem 1. *There is a finite group $G = \langle A, B\rangle$ such that*

$$d(A) = d(B) = 2, \qquad \gcd(|A|, |B|) = 1, \qquad d(G) = 4.$$

Consequently Problem 16.53 has a negative answer for $d = 2$.

Put $S = \mathrm{SL}_2(\mathbb{F}_{11})/\{I, -I\}$. Represent a class by the lexicographically smaller of $(\alpha, \beta, \gamma, \delta)$ and its negative modulo 11, where the entries belong to $\{0, \dots, 10\}$ and $\alpha\delta - \beta\gamma = 1$ modulo 11. Multiplication is matrix multiplication modulo 11 followed by sign normalization. There are $11^2 - 1$ nonzero first columns and 11 second columns of determinant one. Each sign class has two members, so $|S| = 660$.

Define

$$X = \{(x_1, x_2) \in S^2 : x_1x_2 \ne x_2x_1,\ |\langle x_1, x_2\rangle| \in \{6, 12\}\},$$
$$Y = \{(y_1, y_2) \in S^2 : |\langle y_1, y_2\rangle| = 55\}, \qquad Q = X \times Y.$$

For $q = (x_1, x_2, y_1, y_2) \in Q$, define elements of S^Q by

$$a_1(q) = x_1, \quad a_2(q) = x_2, \quad b_1(q) = y_1, \quad b_2(q) = y_2.$$

Finally put

$$A = \langle a_1, a_2\rangle, \qquad B = \langle b_1, b_2\rangle, \qquad G = \langle a_1, a_2, b_1, b_2\rangle \le S^Q.$$

This defines all four generators without enumerating the ambient direct product. The certificate files `matrices.json`, `x_pairs.json`, and `y_pairs.json` contain sorted lists with zero-based indices. A row (i, j) of the first pair list and a row (k, l) of the second specify the coordinate whose entries are the corresponding four matrix classes.

2. The finite certificate

The exhaustive matrix computation gives $|X| = 9240$ and $|Y| = 31680$. Table 1 records its subgroup breakdown. In the final column, an entry $r : s$ means that exactly s elements have order r. Every ordered pair generates a unique subgroup, so the indicated products count pairs without repetition. In particular,

$$\begin{aligned} |Q| &= (110 \cdot 18 + 55 \cdot 36 + 55 \cdot 96)(12 \cdot 2640) \\ &= 292\,723\,200 > 287\,496\,000 = 660^3. \end{aligned}$$

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TABLE 1. Exact subgroup and generating-pair counts.

Family	Order	Subgroups	Pairs each	Element-order spectrum
X	6	110	18	$1:1,\ 2:3,\ 3:2$
X	12	55	36	$1:1,\ 2:7,\ 3:2,\ 6:2$
X	12	55	96	$1:1,\ 2:3,\ 3:8$
Y	55	12	2640	$1:1,\ 5:44,\ 11:10$

Exhaustive matrix check. The preserved program `verify_counterexample.py` enumerates the canonical determinant-one matrices and their exact multiplication table. For every eligible ordered pair it closes the identity under right multiplication by the two entries, abandoning a search only after the allowed subgroup order is exceeded. In a finite group the generated monoid is the generated subgroup. The filters requiring entry orders to divide 12 for X and 55 for Y are necessary by Lagrange's theorem. Thus no eligible pair is discarded. The program also checks inverse and product closure of each counted subgroup, distinctness of the pair lists, and noncommutation of both entries of every listed X-pair and Y-pair.

Independent permutation check. The preserved program `independent_check.py` reads no matrix-checker output. It forms the permutation group on $\mathbb{F}_{11} \cup \{\infty\}$ generated by $z \mapsto z+1$ and $z \mapsto -1/z$, encoding ∞ by 11. These maps are the projective actions of $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. The action of S is faithful: fixing ∞ and zero forces $\beta = \gamma = 0$, and fixing 1 then forces the matrix to be scalar. A scalar determinant-one matrix is I or $-I$. The permutation closure has 660 elements and therefore equals the full image of S.

The program finds one subgroup of each listed spectrum and takes its distinct conjugates by the 660 permutations and by $z \mapsto 2z$. It checks directly that the latter permutation normalizes the group. Within each resulting subgroup it counts ordered generating pairs using a separate left-multiplication closure algorithm. The resulting numbers of distinct pairs are again 9240 and 31680. These are sufficient lower bounds for X and Y; no subgroup classification is needed.

3. Proof of the theorem

For $q = (x_1, x_2, y_1, y_2)$ put $H_q = \langle x_1, x_2 \rangle$ and $K_q = \langle y_1, y_2 \rangle$. Then

$$A \le \prod_{q \in Q} H_q, \qquad B \le \prod_{q \in Q} K_q.$$

Each $|H_q|$ is 6 or 12, and each $|K_q|$ is 55. The products are finite, so Lagrange's theorem confines the prime divisors of $|A|$ to $\{2, 3\}$ and those of $|B|$ to $\{5, 11\}$. Their orders are coprime.

The displayed pairs generate A and B and are noncommuting, as checked coordinatewise by the finite certificate. One explicit coordinate is

$$x_1 = \begin{pmatrix} 0 & 1 \\ 10 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 & 1 \\ 10 & 2 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 & 2 \\ 5 & 8 \end{pmatrix}.$$

Its matrix-index pairs are $(0, 79) \in X$ and $(2, 19) \in Y$. Both coordinate images are nonabelian; hence neither A nor B is cyclic, and $d(A) = d(B) = 2$. By definition, $G = \langle A, B \rangle$.

For each $q \in Q$, coordinate projection restricts to a homomorphism $e_q : G \to S$. Different quadruples differ in at least one entry, so their projections disagree on at least one of a_1, a_2, b_1, b_2. Thus these $|Q|$ homomorphisms are distinct.

If G had at most three generators, padding a generating set with identities would give a generating triple (g_1, g_2, g_3). A homomorphism $G \to S$ is uniquely determined by its values on this triple, so there would be at most $|S|^3 = 287\,496\,000$ such homomorphisms. The $292\,723\,200$ coordinate projections contradict that bound. Therefore $d(G) \ge 4$. The four displayed generators give equality, contradicting $d(G) \le d + 1$ when $d = 2$. □

Methods and provenance

The construction and programs come from solver 1 of the archived campaign run identified in `source/state.json`, configured with `gpt-6-astra`. Its validator recorded PASS after replaying both programs. This export preserves those sources and replays both under CPython 3.12.14; all three matrix/pair files match their archived SHA-256 digests. The proof uses elementary finite-group facts and the displayed exact counts.

This is a draft for independent review. The campaign verdict is neither human approval nor the current repository's two-blind-verifier qualification. A GAP checker is supplied but was not executed because GAP was unavailable; the repository-required GAP reproduction remains pending. No novelty audit or human sign-off is claimed. Reproduction instructions, source checksums, and scoped review records accompany the draft.