A NEGATIVE ANSWER TO PROBLEM 17.32 FROM THE KOUROVKA NOTEBOOK

Abstract. Problem 17.32 of the Kourovka Notebook (O. V. Bogopolski) asks whether, for w in the free group F_n of rank n and $\varphi \in \operatorname{Aut} F_n$, the equality $\langle w, \varphi(w), \ldots, \varphi^n(w)\rangle = F_n$ forces $\langle w, \varphi(w), \ldots, \varphi^{n-1}(w)\rangle = F_n$. We answer this in the negative for every $n \geq 2$, by explicit pairs for which the first n orbit elements generate a subgroup of infinite index. The variant with normal closures fails at $n = 2$: there are pairs with $F_2/\langle\langle w, \varphi(w)\rangle\rangle \cong \mathrm{SL}(2, 7)$ while $\langle\langle w, \varphi(w), \varphi^2(w)\rangle\rangle = F_2$. The weaker conclusion that *some* n of the $n+1$ orbit elements generate F_n fails at $n = 3$. The abelianized statement is true by the Cayley–Hamilton theorem, so all counterexamples are invisible in homology.

1. The problem

Problem 17.32 of the Kourovka Notebook, posed by O. V. Bogopolski in the 17th issue (2010), reads as follows [7, Problem 17.32].

> **17.32.** Is the following analogue of the Cayley–Hamilton theorem true for the free group F_n of rank n: If $w \in F_n$ and $\varphi \in \operatorname{Aut} F_n$ are such that $\langle w, \varphi(w), \ldots, \varphi^n(w)\rangle = F_n$, then $\langle w, \varphi(w), \ldots, \varphi^{n-1}(w)\rangle = F_n$?
>
> *O. V. Bogopolski*

Throughout the Notebook a bare $\langle \ldots \rangle$ denotes the generated subgroup — a normal closure is always named in words and written with an exponent, "the normal closure $\langle x\rangle^G$ of x in G" [7] — so we answer the problem as printed. Its statement is word-for-word identical in issues No. 18 [10], No. 20 [6] and No. 21 [7], carries no star or comment, and is absent from the Archive of Solved Problems. Since the Cayley–Hamilton analogy might nevertheless be drawn with conjugation allowed, we also settle the two natural variants: the *normal-closure variant*, with both brackets read as normal closures $\langle\langle \cdot \rangle\rangle$, and the *subset variant*, whose conclusion asks only that some n of the $n+1$ listed elements generate F_n. Here $\varphi^0 = \mathrm{id}$, so the hypothesis involves $n+1$ elements and the conclusion the first n of them.

Theorem 1.1. *For every $n \geq 2$ the implication in Problem 17.32 fails: there exist $w \in F_n$ and $\varphi \in \operatorname{Aut} F_n$ such that $\langle w, \varphi(w), \ldots, \varphi^n(w)\rangle = F_n$ while $\langle w, \varphi(w), \ldots, \varphi^{n-1}(w)\rangle$ is a proper subgroup of infinite index in F_n. Hence the answer to Problem 17.32 is negative for every $n \geq 2$.*

Theorem 1.2. *The normal-closure variant fails for $n = 2$: there exist $w \in F_2$ and $\varphi \in \operatorname{Aut} F_2$ with $\langle\langle w, \varphi(w), \varphi^2(w)\rangle\rangle = F_2$ while $\langle\langle w, \varphi(w)\rangle\rangle \neq F_2$ — indeed $F_2/\langle\langle w, \varphi(w)\rangle\rangle \cong \mathrm{SL}(2, 7)$.*

Theorem 1.3. *The subset variant fails for $n = 3$: there exist $w \in F_3$ and $\varphi \in \operatorname{Aut} F_3$ such that $\langle w, \varphi(w), \varphi^2(w), \varphi^3(w)\rangle = F_3$, yet no three of the four elements $w, \varphi(w), \varphi^2(w), \varphi^3(w)$ generate F_3.*

For $n = 1$ the implication is trivially true ($\varphi(w) \in \{w, w^{-1}\}$), and the failure in Theorem 1.1 is as strong as possible short of finite index. Theorems 1.2 and 1.3 refute the two variants as general principles; the remaining fixed-rank cases — the normal-closure variant for $n \geq 3$, the subset variant at $n = 2$ and at each fixed $n \geq 4$ — stay open (Section 6). The restriction to $n \geq 2$

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is forced by homology: the abelianized statement is exactly the content of the Cayley–Hamilton theorem and is *true*.

Proposition 1.4. *Let $M \in \mathrm{GL}_n(\mathbb{Z})$ be the matrix induced by $\varphi \in \operatorname{Aut} F_n$ on $H_1(F_n) = \mathbb{Z}^n$ and let v be the image of $w \in F_n$. If $v, Mv, \ldots, M^n v$ generate $\mathbb{Z}^n$, then $v, Mv, \ldots, M^{n-1}v$ already do.*

Proof. χ_M is monic of degree n over $\mathbb{Z}$, so $M^n v \in \langle v, Mv, \ldots, M^{n-1}v\rangle$ [8, Ch. XIV, §3]. □

Consequently, in any counterexample the first n orbit elements map to a *basis* of $\mathbb{Z}^n$; every failure below is invisible in homology, which is exactly the room the free group has and the abelian group does not. (Our witnesses comply: in Theorem 2.2 the images are $(1, 1)$ and $(2, 1)$.)

Notation. F_n is free of fixed rank n on a fixed basis; $F_2 = F(a, b)$, $F_3 = F(a, b, c)$. Beyond the universal property of free groups we use only: Stallings' folded graphs, for two subgroup computations in Section 4 [12, 5]; the Nielsen–Schreier theorem and Schreier's index formula, $\operatorname{rank} H - 1 = [F_n : H]\,(n - 1)$ for H of finite index [5, Cor. 8.4]; the Hopf property of finitely generated free groups [9, Prop. I.3.5]; and standard facts about $\mathrm{SL}(2, 7)$ [3]. This note was produced by an automated pipeline built on a large language model and verified independently of its authorship; see the final section.

2. Rank two

Let $\varphi\colon a \mapsto ba,\ b \mapsto a$ be the Fibonacci substitution on F_2 and $u = ab$, so that

$$u = ab, \qquad \varphi(u) = baa, \qquad \varphi^2(u) = ababa.$$

Lemma 2.1. *$H := \langle ab, baa\rangle$ contains neither a nor b, and $[F_2 : H] = \infty$.*

Proof. Among the four words ab, baa, $b^{-1}a^{-1}$, $a^{-1}a^{-1}b^{-1}$ the last letters are b, a, a^{-1}, b^{-1} and the first letters are a, b, b^{-1}, a^{-1}, so the only products xy of two of them in which cancellation occurs are the trivial ones $y = x^{-1}$. Hence every nonempty reduced product of factors $(ab)^{\pm 1}$, $(baa)^{\pm 1}$ is already a reduced word of length ≥ 2: $a, b \notin H$, and H is free with basis $\{ab, baa\}$. If H had finite index i, Schreier's formula would give $2 - 1 = i\,(2 - 1)$, so $i = 1$ and $H = F_2 \ni a$ — a contradiction. □

An explicit permutation certificate for Lemma 2.1 — a homomorphism $F_2 \to S_4$ under which ab and baa fix a point that a and b move — is recorded in the repository accompanying this note.

Theorem 2.2. *With $\varphi\colon a \mapsto ba,\ b \mapsto a$ and $u = ab$:*

$$\langle u, \varphi(u), \varphi^2(u)\rangle = \langle ab, baa, ababa\rangle = F_2,$$

while $\langle u, \varphi(u)\rangle = \langle ab, baa\rangle$ is a proper subgroup of infinite index. So the implication of Problem 17.32 fails at $n = 2$.

Proof. φ is invertible, with inverse $a \mapsto b$, $b \mapsto ab^{-1}$. Since $ababa = (ab)^2 a$, the subgroup $\langle ab, ababa\rangle$ contains $a = (ab)^{-2}(ababa)$ and $b = a^{-1}(ab)$, so $\langle ab, baa, ababa\rangle \supseteq \langle ab, ababa\rangle = F_2$. The rest is Lemma 2.1. □

Remark 2.3. This is essentially the smallest counterexample: an exhaustive search over all pairs with $|w| \leq 3$ and φ a product of at most two generators of $\operatorname{Aut} F_2$ (swap, inversion $a \mapsto a^{-1}$, transvection $a \mapsto ab$) finds exactly eight counterexamples, all obtained from this pair by inversion $w \leftrightarrow w^{-1}$, by translating the window along the φ-orbit, and by the mirror symmetry exchanging a and b.

3. All ranks

Fix $n \geq 3$, let F_n be free on $\{a, b, c_1, \ldots, c_{n-2}\}$, and define $\varphi \in \operatorname{End} F_n$ by

$$\varphi(a) = ba, \qquad \varphi(b) = a, \qquad \varphi(c_i) = c_{i+1} \ (1 \leq i \leq n-3), \qquad \varphi(c_{n-2}) = ab\, c_1,$$

with $w = c_1$; for $n = 3$ the third clause is vacuous and the last reads $\varphi(c_1) = abc_1$. Thus φ is Fibonacci on $\langle a, b\rangle$ and cycles the c-letters, re-entering the cycle through the word ab.

Lemma 3.1. *$\varphi \in \operatorname{Aut} F_n$, and the φ-orbit of $w = c_1$ is*

$$\varphi^k(c_1) = c_{1+k} \ (0 \leq k \leq n-3), \qquad \varphi^{n-2}(c_1) = ab\, c_1,$$
$$\varphi^{n-1}(c_1) = baa \cdot \varphi(c_1), \qquad \varphi^n(c_1) = ababa \cdot \varphi^2(c_1).$$

Proof. The endomorphism $\psi\colon a \mapsto b$, $b \mapsto ab^{-1}$, $c_j \mapsto c_{j-1}$ $(2 \leq j \leq n-2)$, $c_1 \mapsto ba^{-1}b^{-1}c_{n-2}$ is a two-sided inverse of φ: on a, b and on the chain letters this is immediate, and the two wrap-around checks reduce freely, $\psi\varphi(c_{n-2}) = b \cdot ab^{-1} \cdot ba^{-1}b^{-1}c_{n-2} = c_{n-2}$ and $\varphi\psi(c_1) = a \cdot (ba)^{-1} \cdot a^{-1} \cdot abc_1 = c_1$. The orbit: the first clause is the definition iterated along the chain, then $\varphi(abc_1) = ba \cdot a \cdot \varphi(c_1)$ and $\varphi(baa \cdot \varphi(c_1)) = a \cdot ba \cdot ba \cdot \varphi^2(c_1)$. □

Here $\varphi(c_1) = c_2$ and $\varphi^2(c_1) = c_3$ when those letters exist; the wrap-around values are $\varphi(c_1) = abc_1$ and $\varphi^2(c_1) = baa \cdot abc_1$ for $n = 3$, and $\varphi^2(c_1) = abc_1$ for $n = 4$.

Theorem 3.2. *For every $n \geq 3$ the pair (w, φ) satisfies $\langle w, \varphi(w), \ldots, \varphi^n(w)\rangle = F_n$, while*

$$\langle w, \varphi(w), \ldots, \varphi^{n-1}(w)\rangle = K := \langle ab, baa, c_1, \ldots, c_{n-2}\rangle$$

is a proper subgroup of infinite index. So the implication of Problem 17.32 fails at every $n \geq 3$.

Proof. Write $H = \langle w, \ldots, \varphi^{n-1}(w)\rangle$. $H \subseteq K$: the orbit elements through step $n-3$ are the letters c_{1+k}, and the next two are $ab\, c_1$ and $baa \cdot \varphi(c_1)$, where $\varphi(c_1)$ — c_2 for $n \geq 4$, abc_1 for $n = 3$ — lies in K. $K \subseteq H$: the letters $c_1, \ldots, c_{n-2}$ are orbit elements, and Lemma 3.1 restates as the exchange identities

$$ab = \varphi^{n-2}(w) \cdot w^{-1}, \qquad baa = \varphi^{n-1}(w) \cdot \varphi(w)^{-1}, \qquad ababa = \varphi^n(w) \cdot \varphi^2(w)^{-1}, \tag{1}$$

the first two of which put $ab, baa \in H$, all four factors involved being orbit elements at steps $\leq n-1$. So $H = K$.

The retraction $\rho\colon F_n \to F(a, b)$ killing every c_i maps K onto $\langle ab, baa\rangle$, so $a \in K$ or $b \in K$ would contradict Lemma 2.1; thus K is proper. Being n-generated, K is free of rank $\leq n$ (Nielsen–Schreier), so finite index i would force $i\,(n-1) \leq n-1$, i.e. $i = 1$ and $K = F_n$; hence $[F_n : K] = \infty$.

Finally, $\langle w, \ldots, \varphi^n(w)\rangle$ contains $K = H$ and $\varphi^n(w)$; as $\varphi^2(w)$ is the orbit element at step $2 \leq n-1$, the third identity of (1) yields $ababa$, hence $a = (ab)^{-2}(ababa)$ and $b = a^{-1}(ab)$, and with the c_i the whole basis. So it is F_n. □

Proof of Theorem 1.1. $n = 2$ is Theorem 2.2; $n \geq 3$ is Theorem 3.2. □

Remark 3.3. The construction is designed so that killing the new letters retracts the trapped subgroup onto the rank-two trap: the orbit walks through the fresh letters first and replays the rank-two orbit in its last three steps. A more naive generalization (φ the cycle $x_1 \mapsto \cdots \mapsto x_n \mapsto x_nx_1$ with $w = x_{n-1}^{-1}x_{n-2}^{-1}x_n$, found by a machine scan at $n = 3, 4$) fails at $n = 5$, where the $n+1$ orbit elements no longer generate. Counterexamples at $n = 3$ are nonetheless plentiful: a bounded search found 77, among them $w = ab\, c^{-1}$ with $\varphi = (\text{Fibonacci} \times \mathrm{id}_c)$.

4. The normal-closure variant

The examples above do not test the normal-closure variant: $F_2/\langle\langle ab, baa\rangle\rangle = \langle a, b \mid ab,\ baa\rangle$ is trivial ($b = a^{-1}$, then $a = 1$), so $\langle\langle u, \varphi(u)\rangle\rangle = F_2$ already, and the family behaves the same way after killing the c_i. The variant fails nonetheless at $n = 2$, by pairs of a different kind.

Theorem 4.1. *Let (w, φ) be either of the pairs*
(C1) $w = a^3b^{-4}$, $\quad \varphi\colon a \mapsto bab^2,\ b \mapsto ab^2$;
(C2) $w = a^{-3}b^{-4}$, $\quad \varphi\colon a \mapsto babab,\ b \mapsto ab$.
Then $\varphi \in \operatorname{Aut} F_2$,

$$Q_1 := F_2/\langle\langle w, \varphi(w)\rangle\rangle \cong \mathrm{SL}(2,7), \qquad Q_2 := F_2/\langle\langle w, \varphi(w), \varphi^2(w)\rangle\rangle = 1.$$

In particular $\langle\langle w, \varphi(w), \varphi^2(w)\rangle\rangle = F_2$ while $\langle\langle w, \varphi(w)\rangle\rangle \neq F_2$, which proves Theorem 1.2.

For (C1), $\varphi(w) = bab^3aba^{-1}b^{-2}a^{-1}b^{-2}a^{-1}$ (length 14) and $|\varphi^2(w)| = 37$; for (C2), $|\varphi(w)| = 23$ and $|\varphi^2(w)| = 76$.

Proof. In both cases $\langle\varphi(a), \varphi(b)\rangle = F_2$ (an immediate Stallings-graph computation [12]), so φ is a surjective endomorphism, hence an automorphism, F_2 being Hopfian [9, Prop. I.3.5].

For (C1) take

$$A_0 = \begin{pmatrix} 5 & 2 \\ 0 & 3 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & 6 \\ 1 & 3 \end{pmatrix} \qquad \in \mathrm{SL}(2,7).$$

Direct 2×2 arithmetic modulo 7 shows that $w(A_0, B_0) = \varphi(w)(A_0, B_0) = I$ — for instance $A_0^3 = -I = B_0^4$, whence $w(A_0, B_0) = A_0^3B_0^{-4} = I$ — and that A_0, B_0 generate $\mathrm{SL}(2,7)$. Hence $Q_1 \twoheadrightarrow \mathrm{SL}(2,7)$. A coset enumeration for $\langle a, b \mid w, \varphi(w)\rangle$ over the trivial subgroup completes with 336 cosets, and a completed enumeration computes the index [2], so $|Q_1| = 336 = |\mathrm{SL}(2,7)|$ and $Q_1 \cong \mathrm{SL}(2,7)$. The image of $\varphi^2(w)$ in Q_1 has order 8, hence is noncentral, the centre $\{\pm I\}$ having order 2; since $\mathrm{SL}(2,7)$ is perfect and $\mathrm{PSL}(2,7)$ is simple [3, Thms. 3.2, 3.4, 3.5], every proper normal subgroup of $\mathrm{SL}(2,7)$ is central, so the normal closure of that image is all of Q_1 and $Q_2 = 1$. For (C2) the argument is identical, with an analogous matrix certificate; the image of $\varphi^2(w)$ in Q_1 then has order 6. $\square$

The completed enumerations are the one machine-dependent point of this note; they were reproduced on three independent implementations, and the transcripts and certificates are preserved in the repository (final section).

The mechanism deserves note: $(w, \varphi(w))$ is a balanced presentation of the quasisimple group $\mathrm{SL}(2,7)$ — deficiency-zero presentations of $\mathrm{SL}(2,p)$ are classical [1]; here an orbit segment produces one — and the next orbit element lands noncentrally in it, so adjoining it as a relator kills everything. As it must, the failure is invisible in homology: conjugation acts trivially on H_1, so Proposition 1.4 applies and Q_1 is accordingly perfect. Neither pair satisfies the hypothesis of the problem as printed — the plain subgroup $\langle w, \varphi(w), \varphi^2(w)\rangle$ is proper (again a Stallings computation) — so these pairs and that of Section 2 refute the two readings disjointly. The pairs were found by a bounded search ($|w| \leq 7$, φ a product of at most five generators of $\operatorname{Aut} F_2$); no minimality is claimed. For every $n \geq 3$ the variant remains open (Section 6).

5. The subset variant

Over $\mathbb{Z}$, when $v, Mv, \ldots, M^nv$ generate, the first n of them already form a basis (Proposition 1.4); the subset variant asks whether some n of the $n+1$ orbit elements must then generate F_n. The rank-two pair satisfies this conclusion, $\langle ab, ababa\rangle = F_2$ being part of the proof of Theorem 2.2. At $n = 3$ the variant fails.

Theorem 5.1. *Let F_3 be free on $\{a,b,c\}$, let $\varphi\colon a\mapsto ba,\ b\mapsto a,\ c\mapsto abc$ and $w=c$ (the $n=3$ member of the family of Section 3). The orbit is*

$$w_0=c,\qquad w_1=abc,\qquad w_2=baaabc,\qquad w_3=abababaaabc.$$

Then $\langle w_0,w_1,w_2,w_3\rangle=F_3$, yet no three of the four elements generate F_3 — indeed a lies in none of the four 3-generated subgroups. This proves Theorem 1.3.

Proof. The four elements are pairwise distinct (lengths $1,3,6,11$), so the 3-element subsets are the four sets D_k (omit w_k); and $\langle w_0,\dots,w_3\rangle=F_3$ is Theorem 3.2 at $n=3$. By Lemma 3.1, $w_2=baa\cdot w_1$ and $w_3=ababa\cdot w_2=(ab)^3\,aa\,w_1$. We show $a\notin D_k$ for each k.

$D_3=\langle c,abc,baaabc\rangle$ is the subgroup $K=\langle ab,baa,c\rangle$ of Theorem 3.2, and $a\notin K$.

$D_2=\langle c,abc,w_3\rangle$ equals $\langle aa,ab,c\rangle$: $ab=w_1w_0^{-1}$ and $aa=(ab)^{-3}w_3w_1^{-1}$; conversely $w_0=c$, $w_1=(ab)c$, $w_3=(ab)^3(aa)(ab)(c)$. The homomorphism $\delta\colon F_3\to\mathbb{Z}/2$ with $\delta(a)=\delta(b)=1$, $\delta(c)=0$ vanishes on D_2 but $\delta(a)=1$. So $a\notin D_2$.

$D_1=\langle c,baaabc,w_3\rangle$ equals $\langle c,baaab,ababa\rangle$: $baaab=w_2w_0^{-1}$ and $ababa=w_3w_2^{-1}$; conversely $w_2=baaab\cdot c$, $w_3=ababa\cdot w_2$. In $H_1(F_3)=\mathbb{Z}^3$ the generators map to $(0,0,1),(3,2,0),(3,2,0)$, and $(1,0,0)$ is not in their span ($3\beta=1$ is impossible). So $a\notin D_1$.

$D_0=\langle abc,baaabc,w_3\rangle$ equals $\langle abc,baa,ababa\rangle$: $baa=w_2w_1^{-1}$ and $ababa=w_3w_2^{-1}$; conversely $w_2=baa\cdot w_1$, $w_3=ababa\cdot baa\cdot w_1$. The homomorphism $\kappa\colon F_3\to F_2$ with $\kappa(a)=a$, $\kappa(b)=b$, $\kappa(c)=b^{-1}a^{-1}$ kills abc, so $\kappa(D_0)=\langle baa,ababa\rangle=\varphi_2\langle ab,baa\rangle$, where φ_2 is the Fibonacci automorphism of Section 2 ($\varphi_2(ab)=baa$, $\varphi_2(baa)=ababa$). If $a\in D_0$ then $\varphi_2^{-1}(a)=b\in\langle ab,baa\rangle$, contradicting Lemma 2.1. So $a\notin D_0$. ☐

The variant is thus false as a general principle but undecided at $n=2$ and at each fixed $n\ge 4$: on the family of Section 3 with $n\ge 4$, dropping $\varphi^{n-1}(w)$ — or $\varphi(w)$ — from the $n+1$ orbit elements leaves a generating set, the identities (1) for ab and $ababa$ surviving both drops and the remaining letters following (machine-checked at $n=4,5,6$); so the subset conclusion holds there.

6. Concluding remarks

Since finitely generated free groups are Hopfian, n elements generate F_n if and only if they form a basis; Problem 17.32 thus asks whether "the orbit prefix of length $n+1$ generates" forces "the orbit prefix of length n is a basis" — true after abelianization (Proposition 1.4), false in F_n itself (Theorem 1.1). The following remain open.

1. *The normal-closure variant for $n\ge 3$.* The mechanism of Section 4 suggests looking for (w,φ) with $F_n/\langle\langle w,\dots,\varphi^{n-1}(w)\rangle\rangle$ quasisimple (or with all proper normal subgroups central) and $\varphi^n(w)$ noncentral in it.
2. *The subset variant at $n=2$ and at each fixed $n\ge 4$* (Section 5).
3. *Smallest counterexamples.* For the problem as printed at $n=2$, the pair of Theorem 2.2 is the only counterexample in the small region covered by the search of Remark 2.3, up to the symmetries listed there; no minimality claim is made beyond that region, nor for the pairs (C1), (C2).
4. Whether (C1) and (C2) are equivalent under the natural action of $\operatorname{Aut} F_2$ on pairs (w,φ) was not determined.

Provenance and verification

This note was produced by an automated research pipeline built on a large language model (`claude-fable-5`), and its claims were verified before assembly: each claim package passed two independent adversarial verification passes, in which every lemma was re-derived and every computation independently reprogrammed from the stated data, together with a citation audit checking every reference against its source. Every computational claim was reproduced on independent implementations — the coset enumerations and quotient identifications in GAP 4.15.1 [13] with the package FGA 1.5.0 [11] and in ACE 3.001 [4], as well as on independently written Todd–Coxeter and Stallings-folding programs, final coset tables statically audited. Complete machine records — scripts, transcripts, verifier reports, and the permutation and matrix certificates mentioned above — are preserved in the repository accompanying this note and are available from the authors.

To our knowledge, the rank-two pair, the suspension family, the subset-variant refutation, and the $\mathrm{SL}(2,7)$ pairs with their collapse mechanism are new here; everything else is classical and cited where used. No published treatment of Problem 17.32 was found in our searches, with the caveat that zbMATH and MathSciNet were not consulted. For $n = 2$ the counterexample coincides, up to a permutation of the generators, with an independent unpublished solution known to us; the extension to all $n \geq 2$ and the results on the two variants appear to be new relative to it.

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