A KAZHDAN GROUP WITH INFINITELY MANY SIMPLE UNITARY QUOTIENTS OF RELATIVE RANK ONE

Abstract. A fixed cocompact arithmetic lattice in $\mathrm{PGL}_3(\mathbb{Q}_5)$ maps onto $\mathrm{PSU}_3(q)$ for every prime $q \equiv 7 \pmod{12}$. Combining the known lattice and reduction theorems with property (T) gives a negative answer to Kourovka Problem 18.109 with Lie rank understood as relative rank.

1. The question and the result

Problem 18.109 of the Kourovka Notebook [5] asks:

> Is it true that a group satisfying Kazhdan's property (T) cannot homomorphically map onto infinitely many simple groups of Lie type of rank 1?
>
> It is known that a group which maps onto $\mathrm{PSL}_2(q)$ for infinitely many q does not have Kazhdan's property (T). A. Jaikin-Zapirain

Here rank means relative, or BN-pair, rank. In particular $\mathrm{PSU}_3(q)$ has relative rank one and absolute root system A_2, of rank two. This is the twisted Lie-rank convention of [4, introduction]. An absolute-rank-one reading concerns the PSL_2 family already covered by the second sentence of the problem.

Theorem 1. *There is a discrete group Γ with Kazhdan's property (T) admitting an epimorphism onto $\mathrm{PSU}_3(q)$ for every prime $q \equiv 7 \pmod{12}$. These quotients include infinitely many isomorphism classes of nonabelian finite simple groups of relative Lie rank one.*

The lattice and its reduction images are due to Evra and Parzanchevski [3]. Property (T) for the ambient local group and its inheritance by lattices are classical [1]. The contribution here is their combination as a counterexample to the stated rank-one quotient assertion. No spectral or Ramanujan assertion enters the proof.

2. The lattice and its quotients

Put $A = \mathbb{Z}[i, 1/5]$, where $i^2 = -1$, and let $*$ denote conjugate transpose. Define

$$\Gamma = \{g \in \mathrm{GL}_3(A) : gg^* = \lambda I_3 \text{ for some } \lambda \in \mathbb{Z}[1/5]^\times\}/A^\times,$$

where units act by scalar matrices. This is $\mathrm{PGU}_3(\mathbb{Z}[1/5])$ for $\mathbb{Q}(i)/\mathbb{Q}$ and the identity Hermitian form, in the notation of [3, §3].

Proof of Theorem 1. By [3, §3, pp. 9–10], Γ is a cocompact lattice in $\mathrm{PGU}_3(\mathbb{Q}_5)$. Since 5 splits in $\mathbb{Q}(i)$, this local group is isomorphic to $\mathrm{PGL}_3(\mathbb{Q}_5)$. The algebraic group PGL_3 is connected, almost $\mathbb{Q}_5$-simple and split of $\mathbb{Q}_5$-rank two. Thus [1, Theorem 1.6.1] gives property (T) for $\mathrm{PGL}_3(\mathbb{Q}_5)$, and [1, Theorem 1.7.1] gives it for the discrete lattice Γ.

Fix a prime $q \equiv 7 \pmod{12}$. Since $q \equiv 3 \pmod 4$, $A/(q) = \mathbb{F}_{q^2}$, and conjugation reduces to $a \mapsto a^q$. Reduction therefore defines

$$\rho_q : \Gamma \longrightarrow \mathrm{PGU}_3(\mathbb{F}_q) = \mathrm{PU}_3(q) = \mathrm{PSU}_3(q).$$

It is well defined on scalar classes: 5 is invertible modulo q and units reduce to nonzero scalars. The first equality follows by rescaling a unitary similitude to an isometry, using the surjective norm $\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$. For the second equality, the determinant quotient is $U_1(\mathbb{F}_q)/U_1(\mathbb{F}_q)^3$, as recorded in

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the proof of [3, Proposition 3.8]; here $U_1(\mathbb{F}_q) = \{a \in \mathbb{F}_{q^2}^\times : a^{q+1} = 1\}$. Its order $q+1$ is coprime to 3, so this quotient is trivial. Proposition 3.8 and Table 3.1 of [3] give a subgroup $\Lambda_5 \leq \Gamma$ whose reduction is exactly $\mathrm{PSU}_3(q)$. The primes $5, q$ are distinct and odd and meet the table's congruence conditions. Hence ρ_q is surjective.

For $q \geq 7$ the group $\mathrm{PSU}_3(q)$ is simple by the unitary simplicity theorem recalled in [2, p. 100]. Its Hermitian space of dimension three has Witt index one: norm-surjectivity gives an isotropic line, whereas a totally isotropic subspace W satisfies $W \subseteq W^\perp$, whence $2\dim W \leq 3$. Thus the isotropic flag building, and hence its BN-pair, has rank one.

Dirichlet's theorem [6, Theorem 18.1] supplies infinitely many primes $q \equiv 7 \pmod{12}$. Their groups cannot have only finitely many isomorphism types. Indeed, in a Witt basis with Gram matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad u = I_3 + tE_{13}, \qquad 0 \neq t \in \mathbb{F}_{q^2}, \quad t + t^q = 0,$$

one has $\det u = 1$ and $uJu^* = J$. Here E_{13} is the matrix unit. Moreover $u^k = I_3 + ktE_{13}$ is scalar precisely when $q \mid k$. Thus its projective image has order q. Nondegenerate Hermitian forms of this dimension over $\mathbb{F}_{q^2}$ are isometric, by Gram–Schmidt and norm-surjectivity, so this gives an element of $\mathrm{PSU}_3(q)$ as defined above. A finite list of finite groups has only finitely many element orders, proving the assertion about isomorphism types. Unitary transvections on opposite isotropic lines do not commute, so the simple groups are nonabelian. ∎

3. Further questions

The construction uses only the unitary family. It does not determine whether a property-(T) group can have infinitely many Suzuki or small Ree simple quotients.

Methods and AI disclosure

This internal candidate was produced with an automated research pipeline using `gpt-6-astra` through Codex. Before assembly, the argument underwent two independent adversarial reviews and a separate source audit. The reviews checked the lattice, reduction maps, rank convention and symbolic identities. No computation is a premise of the proof. The supplied construction is a reduction to the cited classical results; a bounded literature search found no explicit prior solution of this problem, which does not establish priority. Human review remains pending. Source notes and verification records are retained in the accompanying repository under `problems/18.109/`.

References

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