# A FINITE 2-GROUP COUNTEREXAMPLE TO KOUROVKA 18.120

Abstract. We construct a group $P$ of order $2^{32}$ with an exact factorization $P = AB$, where $A$ is abelian and $B$ has class two, but $\langle A, B'\rangle \cap B$ is nonabelian. This gives a negative answer to Problem 18.120, including the additional hypothesis $A \cap B = 1$. This is an unverified draft awaiting verification.

## Problem statement

The Kourovka Notebook, version 45, states:

> 18.120. Let $P = AB$ be a finite $p$-group factorized by an abelian subgroup $A$ and a class-two subgroup $B$. Suppose, if necessary, $A \cap B = 1$. Then is it true that $\langle A, [B, B]\rangle = AB_0$, where $B_0$ is an abelian subgroup of $B$? E. Jabara

We construct a group $P$ of order $2^{32}$ with an exact factorization $P = AB$, where $A$ is abelian and $B$ has class two, but $\langle A, B'\rangle \cap B$ is nonabelian. Use $[g, h] = g^{-1}h^{-1}gh$ and $g^h = h^{-1}gh$ throughout. Products of automorphisms act on the right: $g^{RS} = (g^R)^S$.

No external result is load-bearing beyond elementary presentation, semidirect-product and subgroup facts. The construction and proof below are transcribed from the recorded candidate.

## 1. The envelope

Let $T$ have generators $x, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4, w$. Its power and commutator relations are

$$x^4 = y_2^4 = y_4^4 = 1; \qquad y_1^2 = y_3^2 = 1; \qquad z_i^2 = w^2 = 1;$$
$$[x, y_i] = z_i; \qquad [y_1, y_2] = [y_3, y_4] = w.$$

The $z_i$ and $w$ are central, and all other commutators of the displayed generators are trivial. These relations define the class-two group of order $2^{13}$ with unique normal forms

$$x^{\alpha} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3} y_4^{\beta_4} z_1^{\gamma_1} z_2^{\gamma_2} z_3^{\gamma_3} z_4^{\gamma_4} w^{\gamma_0},$$

where $\alpha, \beta_2, \beta_4$ range over $0, 1, 2, 3$ and the other exponents over $0, 1$.

Let $A_0 = \mathrm{C}_4^4 \times \mathrm{C}_2^4$, with generators $d_1, \ldots, d_4, e_1, \ldots, e_4$ in that order. It acts on $T$ as follows:

$$\begin{aligned}
&x^{d_i} = xy_i; && \\
&y_2^{d_1} = y_2 w; \qquad y_4^{d_3} = y_4 w; && \\
&y_j^{d_i} = y_j && \text{for the other pairs } (i, j); \\
&t^{e_i} = t^{y_i^{-1}} && \text{for every } t \text{ in } T.
\end{aligned}$$

The images of $z_i = [x, y_i]$ and $w$ follow from these formulas. The eight actions commute and respect the stated orders. Define $H_0 = A_0 \ltimes T$ with this conjugation convention. Thus $|H_0| = 2^{25}$.

Put $k_i = e_i y_i$ and $K_0 = \langle x, z_1, z_2, z_3, z_4, w, k_1, k_2, k_3, k_4\rangle$. Here $x, z_i, w$ are central in $K_0$, $[k_1, k_2] = [k_3, k_4] = w$, and the other $k$-commutators are trivial. The orders of $k_1, k_2, k_3, k_4$ are $2, 4, 2, 4$. Normal forms give $|K_0| = 2^{13}$, $A_0 \cap K_0 = 1$, and $H_0 = A_0 K_0$.

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## 2. Two automorphisms

Define $R$ and $S$ to fix $x$. Their images on the $d$ generators are:

| $i$ | $R(d_i)$ | $S(d_i)$ |
|---|---|---|
| 1 | $d_1y_2z_2$ | $d_1y_4z_3z_4$ |
| 2 | $d_2d_3y_3e_3$ | $d_2y_2y_4e_1e_2e_3z_3z_4$ |
| 3 | $d_3$ | $d_3y_1y_2y_3$ |
| 4 | $d_4d_1y_2e_1$ | $d_4y_1y_2e_2e_4$ |

For $F = R, S$ define $F(y_i) = [x, F(d_i)]$. Define

$$R(e_i) = k_iR(y_i)^{-1},$$
$$S(e_1) = k_1S(y_1)^{-1}z_1z_3,$$
$$S(e_2) = k_2S(y_2)^{-1}z_1,$$
$$S(e_3) = k_3S(y_3)^{-1}z_3,$$
$$S(e_4) = k_4S(y_4)^{-1}z_1.$$

These images satisfy all defining relations of $H_0$ and define automorphisms. The finite calculations required below, each on the nine generators $d_1, \dots, d_4, e_1, \dots, e_4, x$, are

$$R^8 = S^4 = 1; \qquad C = [R, S]; \qquad C^4 = 1; \qquad [C, R] = [C, S] = 1.$$

The exact orders are $8, 4, 4$ and $|D| = 128$ for $D = \langle R, S\rangle$. Finiteness and the 2-group property also follow without that order calculation: $D' = \langle C\rangle$ has order dividing 4, and $D/D'$ is a quotient of $\mathrm{C}_8 \times \mathrm{C}_4$.

For checking the class of the second factor, the action on $K_0$ is short:

$$R(k_i) = k_i, \quad R(x) = x, \quad R(w) = w,$$
$$R(z_1) = z_1, \quad R(z_2) = z_2z_3, \quad R(z_3) = z_3, \quad R(z_4) = z_4z_1;$$
$$S(x) = x, \quad S(z_i) = z_i, \quad S(w) = w,$$
$$S(k_1) = k_1z_1z_3, \quad S(k_2) = k_2z_1,$$
$$S(k_3) = k_3z_3, \quad S(k_4) = k_4z_1.$$

Thus $K_0$ is invariant; its action displacements lie in $\langle z_1, z_3, w\rangle$, which is central in $K_0$ and fixed by $D$. Also $C$ fixes $K_0$.

## 3. The factorization and explicit witnesses

Take $P = H_0 \rtimes D$ with $h^F = F(h)$, and view $A = A_0$ and $B = K_0 \rtimes D$ as subgroups. Since $H_0 = A_0K_0$ exactly, $P = AB$ and $A \cap B = 1$. $A$ is abelian. The preceding action formulas and centrality of $C$ show that $B'$ is central in $B$, while $[k_1, k_2] = w \neq 1$ shows that $B$ has class exactly two.

Let $c$ be the element of $D$ corresponding to $C = [R, S]$. Then $c$ belongs to $B'$. For $a$ in $A$ put $t_a = [a, c]$. The following identities hold in $H_0$, with the same identities in $P$:

$$k_1 = e_1t_{d_1}t_{e_3}[t_{d_1}, d_4],$$
$$k_2 = e_2t_{d_2}t_{e_1}t_{e_4}[t_{d_1}, d_4].$$

Both right-hand sides lie in $L = \langle A, B'\rangle$, so $k_1, k_2$ belong to $L \cap B$. Their commutator is the nonidentity element $w$. Consequently $L \cap B$ is nonabelian.

Finally, if $L = AB_0$ for any subgroup $B_0 \leq B$, then $B_0 = L \cap B$: an element $b = ab_0$ of $L \cap B$ has $a = bb_0^{-1}$ in $A \cap B = 1$. Therefore no abelian $B_0$ can satisfy the requested conclusion.

## Scope

No smallest-order claim is made.

## Methods and AI disclosure

The user has classified this campaign result as **unverified**. The automated checks described here are historical records; verification remains pending.

This note was produced by an automated research pipeline built on a large language model (`gpt-6-astra`) using Codex. The claim passed two independent blind adversarial verification passes and a citation audit, which found no external proof citations. Each verifier independently reconstructed the finite group, actions and witness identities in a separately written Python coordinate implementation; a standalone GAP construction supplies independent corroboration of the same group and the exact factorization. The proof is computer-assisted at its finite automorphism and identity checks. Complete scripts, outputs and verifier reports are preserved under `problems/18.120/` in the accompanying repository; the source mapping is `solution/writeup-20260918-notes.md`. The mathematical proof is transcribed without condensation or restructuring. This draft remains an unverified draft awaiting verification.