INVOLUTION CENTRALIZERS AND A LINEAR OBSTRUCTION TO KOUROVKA PROBLEM 18.59

Abstract. There is no group with one nonempty conjugacy class of involutions whose involution centralizers have the form $C_2 \times \mathrm{PSL}_2(P)$, where P is an infinite locally finite field of characteristic two. The proof constructs a transitive linear action on a maximal elementary abelian 2-subgroup and excludes its point stabilizer by a dual-space argument. This gives a negative answer to Kourovka Problem 18.59; periodicity of the ambient group is unnecessary.

1. The problem and the result

Problem 18.59 of D. V. Lytkina asks [1, p. 121]:

> Does there exist a periodic group G such that G contains an involution, all involutions in G are conjugate, and the centralizer of every involution i is isomorphic to $\langle i \rangle \times L_2(P)$, where P is some infinite locally finite field of characteristic 2?

Here periodic means that every element has finite order, $L_2(P) = \mathrm{PSL}_2(P)$, and locally finite means that every finitely generated subfield is finite.

Theorem 1. *No group G has a nonempty single conjugacy class of involutions and involution centralizers isomorphic to $C_2 \times \mathrm{PSL}_2(P)$ for an infinite locally finite field P of characteristic two. In particular, the answer to Problem 18.59 is no.*

All matrix, fusion and linear-action facts needed in the proof are established below; no classification theorem is used. The linear obstruction arose in an internal preliminary analysis of the neighboring Problem 18.60. The present proof derives it in full and reconstructs explicitly the implication from linear actions to Problem 18.59 indicated in the Notebook [1, pp. 121–122].

2. A linear obstruction

Lemma 2. *Let V be an $\mathbb{F}_2$-vector space with $\dim V \geq 4$. There is no subgroup $K \leq \mathrm{GL}(V)$ transitive on $V \setminus \{0\}$ such that, for some nonzero v, the stabilizer K_v preserves a complement W_v to $\langle v \rangle$ and is transitive on $W_v \setminus \{0\}$.*

Proof. Transitivity gives such a complement at every nonzero vector. It is unique. Indeed, another K_v-invariant complement is the graph of a K_v-invariant linear map $\ell : W_v \to \mathbb{F}_2$. Transitivity makes ℓ constant, say ϵ, on $W_v \setminus \{0\}$. Choose distinct nonzero $w_1, w_2 \in W_v$. Then $w_1 + w_2 \neq 0$, and

$$\epsilon = \ell(w_1 + w_2) = \ell(w_1) + \ell(w_2) = 0.$$

Thus $\ell = 0$.

Let $f_v : V \to \mathbb{F}_2$ have kernel W_v and satisfy $f_v(v) = 1$. Uniqueness implies, for $k \in K$,

$$W_{kv} = kW_v, \qquad f_{kv}(kw) = f_v(w). \tag{1}$$

The two K_v-orbits on $V \setminus \{0, v\}$ are $W_v \setminus \{0\}$ and $(v + W_v) \setminus \{v\}$. Hence K has exactly two orbits on ordered pairs of distinct nonzero vectors, distinguished by the value $f_v(w) \in \{0, 1\}$.

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Coordinate reversal commutes with K and therefore either preserves both orbits or interchanges them. Accordingly, one of the following identities holds for every pair of distinct nonzero vectors:

$$f_v(w) = f_w(v), \tag{2}$$

or

$$f_v(w) + f_w(v) = 1. \tag{3}$$

Suppose first that (2) holds. Put $f_0 = 0$. Then $B(v, w) = f_v(w)$ is symmetric, including on the diagonal and at zero. It is linear in its second argument, so symmetry makes it bilinear. For independent $x, y \in V$,

$$B(x + y, x + y) = B(x, x) + B(x, y) + B(y, x) + B(y, y) = 1 + 1 = 0,$$

contrary to $f_{x+y}(x + y) = 1$.

Suppose instead that (3) holds. Fix independent x, y and put $T = \langle x, y\rangle$. If $z \notin T$, all pairs in the following calculation are distinct and nonzero:

$$f_{x+y}(z) = 1 + f_z(x + y) = 1 + f_z(x) + f_z(y) = 1 + f_x(z) + f_y(z).$$

Thus the linear functional $L = f_{x+y} + f_x + f_y$ is 1 on $V \setminus T$. Choose z_1, z_2 whose images in V/T are independent, using $\dim V \geq 4$. The three vectors $z_1, z_2, z_1 + z_2$ lie outside T, whereas $L(z_1 + z_2) = L(z_1) + L(z_2) = 0$, a contradiction. □

3. Local structure and fusion

Suppose that G satisfies the hypotheses of Theorem 1, and fix an involution i. Since P has characteristic two, $\mathrm{PSL}_2(P) = \mathrm{SL}_2(P)$: a scalar matrix of determinant one is the identity. Moreover, $\mathrm{SL}_2(P)$ has trivial centre, as also follows from the centralizer computation below. Thus the unique nonidentity central element in the given direct product is the image of i, and we may write

$$C_G(i) = \langle i\rangle \times S, \qquad S = \mathrm{SL}_2(P).$$

Every element of P belongs to a finite subfield. Squaring is bijective on each finite subfield, so P is perfect. Set

$$u(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad U = \{u(a) : a \in P\}, \qquad A = \langle i\rangle \times U.$$

Every nonidentity involution $h \in S$ satisfies $(h - I_2)^2 = 0$. The map $h - I_2$ has rank one and a one-dimensional kernel. Choose a nonzero fixed vector and complete it to a basis of determinant one. In this basis $h = u(a)$ for some $a \neq 0$; hence the conjugating matrix lies in S.

For $a \neq 0$, commutation of $\left(\begin{smallmatrix} r & b \\ c & d \end{smallmatrix}\right)$ with $u(a)$ gives $c = 0$ and $r = d$; determinant one then gives $r^2 = 1$. In characteristic two this forces $r = d = 1$. Therefore

$$C_S(u(a)) = U \qquad (a \neq 0). \tag{4}$$

In particular, a central element of S lies in U and also commutes with $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, forcing it to be the identity. This verifies the assertion about the centre.

Any elementary abelian 2-subgroup of S containing $u(a)$, $a \neq 0$, lies in U by (4). Conjugating one of its nonidentity elements therefore shows that every nontrivial elementary abelian 2-subgroup of S lies in a conjugate of U. Projecting to S now shows that the maximal elementary abelian 2-subgroups of $C_G(i)$ are exactly

$$\langle i\rangle \times sUs^{-1}, \qquad s \in S.$$

Indeed, an elementary abelian subgroup with nontrivial projection lies in one of these groups, and one with trivial projection lies in $\langle i\rangle$, which is not maximal. Each displayed group is maximal by (4). In particular these maximal subgroups are conjugate within $C_G(i)$. The same conclusion holds in $C_G(j)$ for every involution j, by the assumed conjugacy of involutions.

Since $i \in A$, equation (4) gives

$$C_G(A) = C_{C_G(i)}(A) = \langle i \rangle \times U = A. \tag{5}$$

Thus A is maximal elementary abelian in G: any elementary abelian overgroup would centralize A. For $j \in A \setminus \{1\}$, choose $t_0 \in G$ with $t_0 i t_0^{-1} = j$. Both A and $t_0 A t_0^{-1}$ contain j and are maximal elementary abelian subgroups of $C_G(j)$, since each is already maximal in G. Local conjugacy supplies $\kappa \in C_G(j)$ such that

$$\kappa t_0 A t_0^{-1} \kappa^{-1} = A.$$

Then κt_0 normalizes A and sends i to j. Consequently $N = N_G(A)$ acts transitively on $A \setminus \{1\}$.

4. The stabilizer and the contradiction

Regard A as an additive $\mathbb{F}_2$-vector space V, with i as a nonzero vector. Its conjugation action identifies N/A with a subgroup $K \leq \mathrm{GL}(V)$, by (5). This subgroup is transitive on nonzero vectors. The inverse image in N of its i-stabilizer is exactly $N \cap C_G(i)$, whence

$$K_i = N_{C_G(i)}(A)/A = (\langle i \rangle \times N_S(U))/(\langle i \rangle \times U). \tag{6}$$

The common fixed line of U on P^2 is $\langle (1,0) \rangle$. Every element normalizing U preserves this line and is upper triangular; conversely, multiplication shows that all upper triangular matrices of determinant one normalize U. Explicitly,

$$N_S(U) = \left\{ \begin{pmatrix} r & b \\ 0 & r^{-1} \end{pmatrix} : r \in P^*,\ b \in P \right\}, \qquad \begin{pmatrix} r & b \\ 0 & r^{-1} \end{pmatrix} u(a) \begin{pmatrix} r & b \\ 0 & r^{-1} \end{pmatrix}^{-1} = u(r^2 a).$$

Recording the diagonal entry r identifies $N_S(U)/U$ with P^*. Since squaring is bijective on P^*, this quotient acts transitively on $U \setminus \{1\}$. Thus (6) fixes $\langle i \rangle$, preserves its complement U in A, and acts transitively on the nonzero vectors of that complement.

The field P is infinite, so it has infinite dimension over $\mathbb{F}_2$; hence $\dim_{\mathbb{F}_2} V \geq 4$. All hypotheses of Lemma 2 hold, giving a contradiction. This proves Theorem 1. At no point was periodicity of G used.

Open questions

The assumption that all involutions are conjugate supplies the transitive normalizer action. This argument does not settle the corresponding existence question when that assumption is removed.

Methods and AI disclosure

This internal candidate was developed and typeset in Codex using `gpt-6-astra`. The linear argument reuses an internal preliminary analysis of Problem 18.60; the reduction and the full proof were then checked independently. Before typesetting, two adversarial reviewers, each given the statement and candidate proof without the other review, reconstructed the argument. A source audit authenticated the problem and its neighboring cross-reference; no external theorem is a premise. Independent Python and GAP implementations corroborated nine finite matrix and binary-linear checks, which are not assumptions of the proof. Searches found no exact published solution, without establishing priority. Source notes, reviews, scripts and transcripts are retained in the accompanying Kourovka repository under `problems/18.59/`. Human mathematical acceptance remains pending.

References

1. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory. The Kourovka Notebook*, 2026, arXiv:1401.0300v45, `https://arxiv.org/abs/1401.0300v45`.