# NEAR FRATTINI SUBGROUPS AND KOUROVKA PROBLEM 19.3

Abstract. For proper amalgams, parts (a), (b), (c), (e), and (f) of Kourovka Problem 19.3 are affirmative. Part (d) is false: a countable proper amalgam has $\lambda(G) = 1$ and $\mu(G) = H \neq 1$. If collapsed amalgams are admitted, (a)–(d) and (f) are false, while (e) remains true. The printed hypotheses of (g) have no instances; no inferred intended repair is declared resolved. These are internally verified candidates awaiting human review.

## 1. The questions and the result

Problem 19.3 in the Kourovka Notebook [2] reads:

> Let $G = A *_H B$ be the generalized free product of groups $A$ and $B$ with amalgamated subgroup $H$. Is $\psi(G) = 1$ in the following cases?
> (a) $H$ is finite cyclic and $H_G = 1$.
> (b) $H$ is finite cyclic and either $\lambda(A) \cap H_G = 1$ or $\lambda(B) \cap H_G = 1$.
> (c) $H_G = 1$ and $H$ satisfies the minimum condition on subgroups.
> (d) $\lambda(G) \cap H = 1$.
> (e) $A$ and $B$ are free groups, $H$ is finitely generated and at least one of $|A : H|$ or $|B : H|$ is infinite.
> (f) $H$ is infinite cyclic and is a retract of $A$ and $B$.
> (g) $G$ nearly splits over $H$, and $H$ is a normal subgroup of $G$ of prime order.
>
> M. K. Azarian

The printed statement does not require $H < A, B$. We call the amalgam *proper* when both inclusions are proper and *collapsed* when one is an equality. This distinction affects the answers to (a)–(c) and (f).

An element $g \in G$ is a *non-near generator* if, for every subgroup $S \leq G$, finite index of $\langle S, g\rangle$ in $G$ implies finite index of $S$ in $G$. These elements form $\lambda(G)$. A subgroup $M$ is *nearly maximal* if $[G : M] = \infty$ and every subgroup $M < V \leq G$ has finite index. Their intersection is $\mu(G)$, with intersection $G$ when there are none. The notation $\psi(G)$ is defined when $\lambda(G) = \mu(G)$; thus $\psi(G) = 1$ means that both subgroups are trivial [1, §2]. We shall use $\lambda(G) \leq \mu(G)$: if $g \notin M$ for a nearly maximal $M$, then $\langle M, g\rangle$ has finite index but $M$ does not. Write $H^c = c^{-1}Hc$.

Write $C = \mathrm{core}_G(H)$, the largest normal subgroup of $G$ contained in $H$. This is the Notebook's $H_G$, not the normal closure. For proper amalgams, (a)–(c) are affirmative by a published upper-subgroup theorem and the reductions below. The counterexample to (d) has $[A : H] = [B : H] = 2$ and $\lambda(G) = 1$, $\mu(G) = H \neq 1$. The printed near-splitting definition makes (g) vacuous, as proved in the final mathematical section. The results for (e) and (f) are:

**Theorem 1.** *In case (e), $\lambda(G) = \mu(G) = 1$, without a restriction on the ranks of the free groups and also for collapsed amalgams. In case (f), the same conclusion holds for proper amalgams. If collapsed amalgams are admitted in (f), the assertion is false: $A = H = \mathbb{Z}$ and $B = \mathbb{Z} \times C_2$, with the standard inclusions, give $\lambda(G) = \mu(G) \cong C_2$.*

The external inputs for (a)–(c) are Allenby's upper-containment theorem as restated by Azarian. The proof for (d) is self-contained. For (e) and (f), the external inputs are Azarian's three vanishing criteria, stated below. The finite-cover completion used for (e) is the classical free-factor construction, proved here. The reductions to the criteria, including the arbitrary-rank and

*Date*: Draft — 18 September 2026.

collapsed cases, give those answers. The positive arguments are reductions to existing stronger theorems; no new upper-containment or vanishing theorem is claimed.

## 2. Cases (a), (b), and (c)

**Published input.** For a proper amalgam $G = A *_H B$ whose edge subgroup $H$ satisfies the minimum condition on subgroups, $\mu(G) \leq H$. This is Allenby's theorem, restated as Theorem 3.15(ii), p. 9, of Mohammad K. Azarian, *Conjectures and Questions Regarding Near Frattini Subgroups of Generalized Free Products of Groups* [1]. The source has no blanket cardinality or finite-generation restriction. This is the sole load-bearing external theorem in the positive proofs of (a)–(c).

**A finite-normal-subgroup observation.** If $F_0$ is finite and normal in a group $\mathcal{P}$, then $F_0 \leq \lambda(\mathcal{P})$. Indeed, for $\xi \in F_0$ and $S \leq \mathcal{P}$, the group $\langle S, \xi \rangle$ is contained in $SF_0$ and $[SF_0 : S] = [F_0 : F_0 \cap S] \leq |F_0|$. If $[\mathcal{P} : \langle S, \xi \rangle] < \infty$, then $[\mathcal{P} : SF_0] < \infty$, so $[\mathcal{P} : S] < \infty$. This is exactly the non-near-generator condition.

**Proofs of the proper cases. (c).** The published input gives $\mu(G) \leq H$. Since $\mu(G)$ is characteristic in $G$, it is normal in $G$, hence $\mu(G) \leq C = 1$. As $\lambda(G) \leq \mu(G)$, both are trivial.

**(a).** A finite cyclic group satisfies the minimum condition; apply (c). In fact cyclicity is unnecessary: the same proof works for every finite $H$.

**(b).** $C$ is finite and normal in both $A$ and $B$. The observation gives $C \leq \lambda(A)$ and $C \leq \lambda(B)$. Consequently either intersection condition in (b) forces $C = 1$, and (a) applies. Again cyclicity is unnecessary: the same argument applies to any finite $H$.

These are reductions to an existing stronger upper-subgroup theorem; no claim of a new upper-containment theorem is made.

**Literal collapsed-amalgam cases (a)–(d).** Let $A = H = 1$ and $B = C_2$. The amalgam is $G = C_2$. Every subgroup has finite index, so every element is a non-near generator: $\lambda(G) = G$. There are no nearly maximal subgroups, and their empty intersection is $G$; thus $\mu(G) = G$ and $\psi(G) = C_2 \neq 1$.

Here $H$ is finite cyclic, core-free, and satisfies the minimum condition; $\lambda(A) = 1$ and $\lambda(G) \cap H = 1$. All four printed hypotheses (a)–(d) hold. This example distinguishes the properness convention and is not substituted for an intended-case answer to (d).

## 3. A proper-amalgam counterexample to (d)

For a group $\mathcal{X}$, an element $\xi$ is a near generator if some subgroup $S$ has $[\mathcal{X} : S] = \infty$ but $[\mathcal{X} : \langle S, \xi \rangle] < \infty$. Write $\lambda(\mathcal{X})$ for the non-near generators, and $\mu(\mathcal{X})$ for the intersection of the nearly maximal subgroups ($\mathcal{X}$ if there are none). A subgroup is nearly maximal if it has infinite index and all its proper overgroups have finite index. The group $\psi(\mathcal{X})$ exists only when $\lambda(\mathcal{X}) = \mu(\mathcal{X})$.

**Claim.** For every prime $p$ there is a countable proper amalgam $G = A *_H B$ with $[A : H] = [B : H] = 2$ and

$$\lambda(G) = 1, \qquad \mu(G) = H \neq 1.$$

Thus the condition in 19.3(d) holds, but its conclusion fails.

**Construction.** Let $E$ be the additive group of a countably infinite-dimensional $\mathbb{F}_p$-vector space. Let $R = \mathbb{F}_p[E]$ be its group algebra, let $\mathcal{V} = (R, +)$, and let $W = \mathcal{V} \rtimes E$, where $E$ acts by left multiplication on $R$. Equivalently $W = C_p \wr E$ is the restricted regular wreath product. Put $D = C_2 * C_2$ and

$$A = W \times C_2, \qquad B = W \times C_2, \qquad H = W.$$

The direct-factor embeddings of $H$ in $A$ and $B$ have index 2. Their amalgam is $G = W \times D$: this follows immediately from the presentations, since both generators of $D$ commute with $W$ and no further relation between them is imposed. We identify $W$ and $\mathcal{V}$ with their copies in $G$.

**Elementary facts used.** The non-near generators form a characteristic subgroup: inverses preserve the defining property; if $\xi, \upsilon$ have it, finite index of $\langle S, \xi\upsilon\rangle$ implies finite index of $\langle S, \xi, \upsilon\rangle$, then of $\langle S, \xi\rangle$, then of $S$. Automorphisms preserve all these indices. Also $\lambda(\mathcal{X}) \leq \mu(\mathcal{X})$, since a nearly maximal subgroup excluding $\xi$ witnesses that $\xi$ is near.

If $\pi : \mathcal{X} \to \mathcal{Y}$ is onto, $\pi(\lambda(\mathcal{X})) \leq \lambda(\mathcal{Y})$: the inverse image of a near-generator witness for $\pi(\xi)$ is one for $\xi$. Likewise $\pi(\mu(\mathcal{X})) \leq \mu(\mathcal{Y})$, since inverse images of nearly maximal subgroups are nearly maximal. A finite normal subgroup $N_0$ of $\mathcal{X}$ lies in $\lambda(\mathcal{X})$: if $\langle S, \nu\rangle$ has finite index, then $SN_0$ has finite index and $[SN_0 : S] \leq |N_0|$, so $S$ also has finite index.

In $D$ every reflection subgroup is nearly maximal: adjoining any element outside it gives an infinite subgroup of the infinite dihedral group, hence a finite-index subgroup. These reflection subgroups have trivial intersection. Thus $\mu(D) = \lambda(D) = 1$.

**The lower subgroup.** Fix $0 \neq v \in \mathcal{V}$. The support of $v$ in $E$ lies in a finite-dimensional subgroup $E_1 \leq E$. Choose a vector-space complement $E_0$, so $E = E_1 \oplus E_0$; the complement $E_0$ is infinite. Put $R_0 = \mathbb{F}_p[E_0]$. As an $R_0$-module, $\mathcal{V}$ is free with basis $E_1$. In this basis the coefficients of $v$ lie in $\mathbb{F}_p$ and some coefficient $\gamma$ is nonzero. The corresponding coordinate projection, multiplied by $\gamma^{-1}$, is an $R_0$-linear map

$$\ell : \mathcal{V} \to R_0, \qquad \ell(v) = 1.$$

Let $\mathcal{T}_v = \ker \ell$ and $S = \mathcal{T}_v \rtimes E_0$. The normal closure of $v$ under $E_0$ generates the additive $R_0$-module $R_0 v$, and $\mathcal{T}_v + R_0 v = \mathcal{V}$. Therefore

$$\langle S, v\rangle = \mathcal{V} \rtimes E_0,$$
$$[W : \langle S, v\rangle] = |E_1| < \infty, \qquad [\mathcal{V} \rtimes E_0 : S] = |R_0| = \infty.$$

Hence $v$ is a near generator of $W$, and $\lambda(W) \cap \mathcal{V} = 1$. Normality of $\lambda(W)$ gives $[\lambda(W), \mathcal{V}] = 1$. But $C_W(\mathcal{V}) = \mathcal{V}$: a nonidentity element of $E$ moves the basis element 1 of $R$. It follows that $\lambda(W) \leq \mathcal{V}$ and $\lambda(W) = 1$. The two projections of $G = W \times D$ now give $\lambda(G) \leq \lambda(W) \times \lambda(D) = 1$.

**The upper subgroup.** Suppose $M$ is nearly maximal in $G$ and $\mathcal{V}$ is not contained in $M$. The group $\mathcal{V}M$ is then a proper overgroup of $M$, so $[G : \mathcal{V}M] < \infty$. Set $\mathcal{T}_M = M \cap \mathcal{V}$ and $Q = \mathcal{V}/\mathcal{T}_M$. This is an $M$-module over $\mathbb{F}_p$, and it has infinite dimension because $|Q| = [\mathcal{V}M : M] = \infty$.

If $U$ is a nonzero $M$-invariant subspace of $Q$, let $\mathcal{V}_U$ be its preimage in $\mathcal{V}$. The subgroup $M\mathcal{V}_U$ properly contains $M$, so it has finite index in $G$. The index formula gives

$$\dim_{\mathbb{F}_p}(Q/U) < \infty. \tag{1}$$

Conjugation by $M$ on $\mathcal{V}$ factors through its projection to $E$. Thus all its action operators commute, and each $e$ satisfies $(e-1)^p = 0$. Let $a = e-1$ for such an operator. The image $aQ$ is $M$-invariant. If it is nonzero, (1) makes $Q/aQ$ finite-dimensional. Choose a finite-dimensional subspace $\mathcal{L}$ with $Q = \mathcal{L} + aQ$. Iterating this equality $p$ times and using $a^p = 0$ gives

$$Q = \mathcal{L} + a\mathcal{L} + \cdots + a^{p-1}\mathcal{L},$$

contrary to the infinite dimension of $Q$. Hence $aQ = 0$ for every operator. The action of $M$ on $Q$ is trivial. Any one-dimensional subspace of $Q$ is now invariant and has infinite codimension, contradicting (1). This proves that every nearly maximal subgroup of $G$ contains $\mathcal{V}$, so $\mathcal{V} \leq \mu(G)$.

For the exact value, all nearly maximal subgroups therefore correspond under quotient by $\mathcal{V}$ to those of $G/\mathcal{V} = E \times D$. Every element of $E$ lies in a finite central subgroup of $E \times D$, so $E \leq \lambda(E \times D) \leq \mu(E \times D)$. Projection to $D$ gives the reverse inclusion $\mu(E \times D) \leq E$. Consequently $\mu(G)/\mathcal{V} = E$ and $\mu(G) = W = H$. This is nontrivial, whereas $\lambda(G) = 1$. $\square$

## 4. Vanishing criteria and a free-group lemma

We use the following results of [1, Theorems 3.1 and 3.3, p. 5]:

(1) A free product of two nontrivial groups has $\lambda = \mu = 1$.
(2) For a proper amalgam $G = A *_H B$, if some $c \in G \setminus H$ satisfies $H^c \cap H = 1$, then $\lambda(G) = \mu(G) = 1$.
(3) For a proper amalgam $G = A *_H B$, if $\mu(G) \cap H = 1$, then $\lambda(G) = \mu(G) = 1$.

Only the proper-amalgam instances of the latter two criteria are used.

**Lemma 2.** *If $H$ is a finitely generated infinite-index subgroup of a free group $A$, then $H^c \cap H = 1$ for some $c \in A \setminus H$.*

*Proof.* Choose a finite-rank free factor $F \leq A$ containing $H$ with $[F : H] = \infty$. If $A$ has finite rank, take $F = A$. Otherwise take the finite support of generators of $H$ in a free basis of $A$ and add one unused basis element. The homomorphism $F \to \mathbb{Z}$ sending this extra element to 1 and the others to 0 kills $H$, so $[F : H] = \infty$.

Let $X$ be a finite free basis of $F$. In the Schreier graph of $H$ in $F$, take the finite connected based subgraph $\Gamma$ consisting of the closed paths spelling generators of $H$ and their inverses. Include the base vertex also when $H = 1$. Labels of based loops in $\Gamma$ give exactly $H$. The label map on its fundamental group is injective: a reduced graph path has a freely reduced label because $\Gamma$ is a subgraph of a Schreier graph.

For each $x \in X$, the numbers of vertices missing an outgoing and an incoming $x$-edge are equal. Pair these missing slots and add the corresponding edges. Doing this for all $x$ gives a connected finite cover $\Delta$ of the rose on $X$, on the same vertex set. At least one edge is added; otherwise $\Gamma$ itself would be a finite cover representing $H$, contrary to $[F : H] = \infty$.

A spanning tree $\tau$ of $\Gamma$ also spans $\Delta$. The non-tree edges of $\Gamma$ give a basis of $\pi_1(\Gamma)$, and the new edges give additional basis elements of $\pi_1(\Delta)$. Under the injective label map into $F$, its image therefore splits as $K = H * L$ with $L \neq 1$. Any $1 \neq c \in L$ lies outside $H$, and free-product normal form gives $H \cap c^{-1}Hc = 1$. □

## 5. Proof of the answers

*Proof of Theorem 1.* In case (e), first suppose $H < A, B$. Interchanging the factors if needed, assume $[A : H] = \infty$. Lemma 2 gives $c \in A \setminus H$ with $H^c \cap H = 1$, so the second vanishing criterion gives the conclusion.

If $H = A$ or $H = B$, the amalgam is the other free group $P$. The infinite-index hypothesis ensures $P \neq 1$. If $P$ has rank one, its trivial subgroup is nearly maximal, so $\mu(P) = 1$ and $\lambda(P) = 1$. If $P$ has rank at least two, partition a free basis into two nonempty sets. The first vanishing criterion applies to the resulting free product, also when the rank is infinite.

For a proper amalgam in (f), let $r_A : A \to H$ and $r_B : B \to H$ be the retractions. They agree on $H$, so induce a retraction $r : G \to H$. Put $N = \ker r$. Then $G/N \cong \mathbb{Z}$, so $[G : N] = \infty$. If $N < V \leq G$, the nontrivial subgroup $V/N$ of $G/N$ has finite index. Hence $N$ is nearly maximal and $\mu(G) \leq N$. Since $N \cap H = 1$, the third vanishing criterion proves the claim.

Finally, consider the collapsed amalgam with $A = H = \langle h \rangle \cong \mathbb{Z}$ and $B = \langle h \rangle \times \langle t \rangle$, where $t$ has order two. The identity on $A$ and the projection on $B$ are retractions onto $H$, and $G = B$. Put $T = \langle t \rangle$. Every subgroup $S \leq G$ with nonzero projection onto $\langle h \rangle$ has finite index: if that projection is $\langle h^n \rangle$, $n \geq 1$, then

$$[G : S] = n[T : S \cap T] \leq 2n.$$

Thus the infinite-index subgroups are exactly the subgroups of $T$, and $T$ is the unique nearly maximal subgroup. Consequently $\mu(G) = T$. Also $[\langle S, t \rangle : S] \leq 2$ for every $S \leq G$, so finite index of $\langle S, t \rangle$ implies finite index of $S$. Hence $t \in \lambda(G)$ and $\lambda(G) = \mu(G) = T \neq 1$. □

## 6. The empty hypothesis in (g)

The adjacent Notebook definition says that $G$ nearly splits over a normal subgroup $H$ if there exists $N_* \le G$ with $[G : N_*] = \infty$, $[G : HN_*] < \infty$, and $\mathrm{core}_G(H \cap N_*) = 1$. If $H$ is finite, normality makes $HN_*$ a subgroup and

$$[G : N_*] = [G : HN_*][HN_* : N_*] = [G : HN_*][H : H \cap N_*] < \infty,$$

a contradiction. Thus no group nearly splits over any finite normal $H$ under this definition. Primality, amalgamation, and the intersection-core condition are unnecessary for this conclusion. The universal implication in (g) is therefore vacuous under the literal wording, in both conventions.

A repaired intended version requires clarification and is not declared resolved.

## Methods and AI disclosure

This internal candidate was produced with an automated research pipeline using `gpt-6-astra` through Codex. Before assembly, each of the three proof packages passed two independent blind adversarial reviews; the external statements were checked against primary source notes. Finite-cover examples for (e) were corroborated in Python and independently in GAP/FGA; no computation is a proof premise. The cited criteria and the covering construction are classical. The reductions, the self-contained proper counterexample to (d), and the formulation analysis are supplied here; priority remains unconfirmed. The previously reviewed TeX proofs for (e) and (f) are preserved unchanged. Human review remains pending, and no Notebook acceptance is claimed. Source notes, scripts and verification records are retained in the accompanying repository under `problems/19.3/`; the review pairs are `r1c1-v1/v2`, `r2c1a-v1/v2`, and `r2c1b-v1/v2` in its `verification/` directory.

## References


1. Mohammad K. Azarian, *Conjectures and questions regarding near Frattini subgroups of generalized free products of groups*, International Journal of Algebra **5** (2011), no. 1, 1–15.
2. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory. The Kourovka Notebook. No. 21*, Sobolev Institute of Mathematics, Novosibirsk, 2026, arXiv:1401.0300v45, 3 July 2026.