AN ORDER-96 COUNTEREXAMPLE TO KOUROVKA 20.122(B) AND (C-MIN)

Abstract. We exhibit a soluble group of order 96 and three nilpotent subgroups whose inclusion-minimal triple intersections generate a subgroup outside the Fitting subgroup. This gives negative answers to Kourovka 20.122(b) and the inclusion-minimal assertion in (c). The minimum-order assertions (a) and the other part of (c) remain open. This is an unverified draft awaiting verification.

1. Problem and result

The following is Problem 20.122 of the *Kourovka Notebook*, arXiv:1401.0300v45, proposed by V. I. Zenkov.

> For nilpotent subgroups A, B, C of a finite group G, let $\mathrm{Min}_G(A, B, C)$ be the subgroup of A generated by all minimal by inclusion intersections of the form $A \cap B^x \cap C^y$, where $x, y \in G$, and let $\min_G(A, B, C)$ be the subgroup of $\mathrm{Min}_G(A, B, C)$ generated by all intersections of this kind of minimal order.
> (a) Is it true that $\min_G(A, B, C) \leq F(G)$?
> (b) Is it true that $\mathrm{Min}_G(A, B, C) \leq F(G)$?
> (c) The same questions for soluble groups.

Here $F(G)$ is the Fitting subgroup, and $B^x = x^{-1}Bx$. We use (c-Min) for the inclusion-minimal assertion in (c), and (c-min) for its minimum-order assertion.

Theorem 1. *There is a soluble group G of order 96 with nilpotent subgroups A, B, C of orders 32, 16, 16, respectively, such that*

$$\mathrm{Min}_G(A, B, C) = A \not\leq F(G), \qquad \min_G(A, B, C) = F(G), \qquad |F(G)| = 16.$$

Thus 20.122(b) and (c-Min) have negative answers. The construction does not answer (a) or (c-min).

The group itself is classical; the proposed answer is the specified subgroup triple and its intersection calculation. The argument uses only elementary finite-group theory and two-dimensional linear algebra.

2. Construction

Let $U = \mathbb{F}_2^2$, $H = \mathrm{GL}(U)$, and $V = U \oplus U$. Let H act diagonally on V and put $G = V \rtimes H$. Thus $|G| = 16 \cdot 6 = 96$ and G is soluble. Label the three one-dimensional subspaces of U as L_0, L_1, L_2. Let t_i be the unique nonidentity involution of H fixing L_i pointwise. Then

$$\ker(t_i - 1) = \mathrm{im}(t_i - 1) = L_i.$$

Write $T_i = \langle t_i \rangle$ and identify H with the standard complement in G. Define

$$A = V \rtimes T_0,$$
$$B = (U \oplus L_0) \rtimes T_0,$$
$$C = (L_0 \oplus U) \rtimes T_0.$$

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These have orders $32, 16, 16$, respectively, and are 2-groups, hence nilpotent.

3. Fitting subgroup

The normal elementary abelian subgroup V lies in $F(G)$. The quotient $H = S_3$ has no nontrivial normal 2-subgroup, so $O_2(G) = V$. If N is a normal 3-subgroup of G, then $[N, V]$ lies in $N \cap V = 1$. The faithful action of H on V gives $C_G(V) = V$, hence $N = 1$. In a finite nilpotent normal subgroup every Sylow subgroup is characteristic and therefore normal in G. Since $|G|$ has only primes $2, 3$, this proves $F(G) = V$.

4. Complete conjugate family

Put

$$B_i = (U \oplus L_i) \rtimes T_i, \qquad C_j = (L_j \oplus U) \rtimes T_j.$$

Conjugation by H permutes each displayed family transitively. Conjugation by V fixes every B_i and C_i: its displacement on t_i is contained in $\operatorname{im}(t_i - 1)$ on V, namely $L_i \oplus L_i$, a subgroup of both displayed bases. Consequently B has exactly the three conjugates B_i, and C exactly the three conjugates C_j. This accounts for every $x, y \in G$ in the Notebook statement.

For $i = j = 0$,

$$\begin{aligned} I_{00} &= A \cap B_0 \cap C_0 \\ &= (L_0 \oplus L_0) \rtimes T_0 \\ &= (L_0 \oplus L_0) \times T_0, \end{aligned}$$

which has order 8 and contains t_0 outside $F(G)$. For every other pair (i, j),

$$I_{ij} = A \cap B_i \cap C_j = L_j \oplus L_i.$$

Indeed, the common image of the three subgroups in H is $T_0 \cap T_i \cap T_j$, trivial unless $i = j = 0$; inside V their intersection is precisely $L_j \oplus L_i$. These eight subgroups have order 4 and are pairwise distinct.

5. Minimality and the two generated subgroups

Every I_{ij} with $(i, j) \neq (0, 0)$ has at least one coordinate line different from L_0. It is therefore not contained in $I_{00} \cap V = L_0 \oplus L_0$. The eight order-4 intersections are mutually incomparable, and the order-8 intersection cannot be contained in any of them. Thus *all nine* intersections are inclusion-minimal.

The eight order-4 intersections generate V: those with $j = 0$ and $i = 1, 2$ generate $L_0 \oplus U$, and those with $i = 0$ and $j = 1, 2$ generate $U \oplus L_0$. Together with I_{00} they therefore generate A. Hence

$$\begin{gathered} \mathrm{Min}_G(A, B, C) = A, \qquad |A| = 32, \qquad A \not\leq F(G) = V, \\ \min_G(A, B, C) = V = F(G), \qquad |V| = 16. \end{gathered}$$

This is a negative answer to 20.122(b) and the inclusion-minimal question in (c), already for a soluble group. The minimum-order questions (a) and (c-min) remain unanswered by this construction.

6. Explicit permutations

An independent GAP representation on 8 points is

$$\begin{aligned} a &= (1,2)(3,4), & b &= (1,3)(2,4), \\ c &= (5,6)(7,8), & d &= (5,7)(6,8), \\ t &= (1,2)(5,6), & r &= (1,2,3)(5,6,7). \end{aligned}$$

Take $G = \langle a,b,c,d,t,r\rangle$, $V = \langle a,b,c,d\rangle$, $A = \langle a,b,c,d,t\rangle$, $B = \langle a,b,c,t\rangle$, $C = \langle a,c,d,t\rangle$. GAP identifies G as `SmallGroup(96,227)`. The identifier is supplementary; the proof uses the explicit construction.

Open questions

The minimum-order assertions 20.122(a) and (c-min) remain open. The present example satisfies their asserted containment with equality.

Methods and AI disclosure

The user has classified this campaign result as **unverified**. The automated checks described here are historical records; verification remains pending.

This note was produced by an automated research pipeline using the large language model `gpt-6-astra` through Codex. Its candidate proof passed two independent blind adversarial verification passes and a citation audit; both reviewers reconstructed the full symbolic argument. Independent GAP corroboration on the explicit permutations completed all 9 conjugate pairs, recovering all 9 distinct, inclusion-minimal intersections and both generated subgroups. The proof does not depend on these computed counts. Scripts, transcripts and reports are preserved under `problems/20.122/` in the accompanying repository, including `verification/r1c1-v1.md`, `verification/r1c1-v2.md` and the script and output `experiments/r1-counterexample/fiber-product-check.g` and `fiber-product-check.out`. The proof text is faithfully converted from the recorded `solution/solution.md`. The bounded novelty search found no exact prior answer; it does not certify originality. This is an unverified draft awaiting verification, with no claim of Notebook acceptance.