# AN EXPLICIT COUNTEREXAMPLE TO KOUROVKA NOTEBOOK PROBLEM 20.30

Abstract. This imported construction claims a perfect centreless finite group $G$ with $|G| = 120 \cdot 5^{68}$ and maximum conjugacy-class size $n = 30 \cdot 5^{32}$, so that $|G|/n^2 = 250/3 > 1$. The source argument is reproduced in full below, with changes confined to notation and typesetting. This is an import draft; repository mathematical verification, citation and novelty checks, and independent GAP corroboration remain pending.

## Problem statement

Problem 20.30 (P. M. Neumann and M. R. Vaughan-Lee) asks:

> Let $G$ be a perfect and centreless finite group, and let $n$ be the maximum size of a conjugate class in $G$. Is it true that $|G| \le n^2$?

The Notebook entry is signed I. B. Gorshkov. The statement is transcribed from the repository's copy of the Kourovka Notebook, arXiv:1401.0300v45.

All vector spaces and scalar coefficients below are over $\mathbb{F}_5$. The integer cardinalities and inequalities use ordinary integers. The construction has

$$|G| = 120 \cdot 5^{68}, \qquad \max\{|x^G| : x \in G\} = 30 \cdot 5^{32}, \qquad \frac{|G|}{(\max|x^G|)^2} = \frac{250}{3} > 1.$$

## 1. The finite group, specified without a database identifier

Let $E$ have basis $a_0, \dots, a_7$. Let $U = \mathbb{F}_5^2$, with its standard alternating form

$$\omega((\rho_1, \rho_2), (\sigma_1, \sigma_2)) = \rho_1\sigma_2 - \rho_2\sigma_1.$$

Let $W = \mathrm{Sym}^2(E)$, with basis $a_i a_j$ for $0 \le i \le j < 8$ (no normalization factors in this symmetric-product basis). Let $V = U \otimes E$ and let $Z$ be a separate copy of $U \otimes E$. Thus $\dim V = 16$, $\dim W = 36$ and $\dim Z = 16$.

Define a symmetric trilinear map $T : E^3 \longrightarrow E$ by its values on basis triples:

$$T(a_i, a_j, a_k) = \begin{cases} a_\ell, & \text{if exactly two indices agree and } \ell \text{ is the other index,} \\ 0, & \text{if all three agree or all three are distinct.} \end{cases}$$

On $L = V \oplus W \oplus Z$ define a bilinear, alternating bracket by

$$[\xi \otimes a, \eta \otimes b] = \omega(\xi, \eta)ab \quad \text{in } W, \qquad [\xi \otimes a, bc] = \xi \otimes T(a, b, c) \quad \text{in } Z.$$

All brackets involving $Z$ and all brackets within $W$ are zero; reversed brackets have the opposite sign. This specifies the bracket completely.

It is a Lie bracket. Only Jacobi identities on three vectors in $V$ can be nonzero. For $\xi \otimes a, \eta \otimes b, \zeta \otimes c$, their Jacobi sum is, up to the common choice of nesting sign,

$$(\omega(\xi, \eta)\zeta + \omega(\eta, \zeta)\xi + \omega(\zeta, \xi)\eta) \otimes T(a, b, c) = 0.$$

The vector identity in parentheses holds in a two-dimensional alternating space, and $T$ is symmetric. Furthermore $[L, L]$ is contained in $W + Z$ and $[L, [L, L]]$ is contained in the central subspace $Z$. The nilpotency class is at most 3.

Make the set $L$ into a group $N$ with the following explicit multiplication:

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\big([x, [x, y]] + [y, [y, x]]\big).$$

*Date*: Draft.

Here $1/2 = 1/12 = 3$ in $\mathbb{F}_5$. This is the class-three BCH multiplication. Its validity can also be checked directly without invoking a correspondence: expand $(x * y) * z$ and $x * (y * z)$, discard brackets of length at least four, and subtract. The difference is

$$-\frac{1}{6}([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) = 0.$$

Thus it is associative. Its identity is 0, its inverse map is $x \mapsto -x$, and $x$ raised to an integer $r$ is $rx$. In particular $|N| = 5^{68}$ and $N$ has exponent 5. The denominators 2, 6 and 12 are all invertible in $\mathbb{F}_5$.

Let $H = \mathrm{SL}_2(\mathbb{F}_5)$. It acts naturally on the $U$ factor of $V$ and $Z$ and fixes $W$ pointwise. These transformations preserve the bracket: the $V, V$ bracket uses the determinant form, and the $V, W$ bracket is linear in $U$. Consequently they preserve the displayed group multiplication. Define

$$G = N \rtimes H, \qquad (x, h)(y, h') = (x * h(y), hh').$$

This is a complete description of every group element and its product.

There are 120 determinant-one 2-by-2 matrices over $\mathbb{F}_5$, so $|G| = 120 \cdot 5^{68}$.

## 2. Perfectness

First $H$ is perfect. For an entirely elementary check, put

$$u_+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad u_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad d = \mathrm{diag}(2,3).$$

Conjugation by $d$ multiplies the parameter of both $u_+(t)$ and $u_-(t)$ by 4 in $\mathbb{F}_5$. Thus

$$du_+(t)d^{-1}u_+(t)^{-1} = u_+(3t),$$

and the corresponding formula gives $u_-(3t)$. Hence all upper and lower elementary matrices belong to $H'$. They generate $\mathrm{SL}_2(\mathbb{F}_5)$, as row reduction shows, so $H' = H$. The certificate additionally enumerates $H$ and computes $|H'| = 120$.

Let $s = -I$ in $H$. For a pure vector $x$ in $V$ or $Z$, $s(x) = -x$. With the commutator convention $[\alpha, \beta]_{\text{group}} = \alpha^{-1}\beta^{-1}\alpha\beta$, in $G$ we have

$$[(x, 1), (0, s)]_{\text{group}} = (-2x, 1).$$

Since $-2$ is invertible in $\mathbb{F}_5$, every pure $V$ element and every $Z$ element belongs to $G'$. For the standard basis $e, f$ of $U$,

$$[e \otimes a_i, f \otimes a_j]_{\text{Lie}} = a_i a_j.$$

In $N/Z$ the group commutator equals this Lie bracket, since the quotient has class at most two. Therefore the corresponding group commutators give all $W$ basis vectors modulo $Z$. As $Z$ is already in $G'$, every pure $W$ basis vector is in $G'$, and $W$ is additive under the group law. The pure $V$, $W$ and $Z$ elements generate $N$. It follows that $N$ and $H$ are contained in $G'$, proving $G' = G$.

## 3. The centre is trivial

For class-three BCH multiplication one has, as vectors,

$$x * y - y * x = [x, y]_{\text{Lie}}.$$

Thus the centre of $N$ is exactly the centre of $L$. Write $x = v + w + z$ in $L$. If $x$ is central, its brackets with $V$, taken modulo $Z$, force $v = 0$: bracketing with $e \otimes a_j$ or $f \otimes a_j$ reads off the coefficients of $v$ on the linearly independent vectors $a_i a_j$ for any fixed $j$.

Write

$$w = \sum_i d_i a_i^2 + \sum_{i<j} s_{ij} a_i a_j, \qquad D_0 = \sum_i d_i.$$

The map $a \mapsto T(a, w)$ has matrix with off-diagonal entries $s_{ij}$ and diagonal entry $D_0 - d_i$ in position $i$. If this map is zero, all $s_{ij} = 0$ and $d_i = D_0$ for all $i$. Summing gives $D_0 = 8D_0$, hence $7D_0 = 0$. Since $7 = 2$ is nonzero in $\mathbb{F}_5$, all $d_i$ vanish. Therefore $Z(N) = Z$.

Put $A = W + Z$. The quotient $N/A$ is the additive space $V$. If $(x, h)$ is central in $G$, its action on $N/A$ must be trivial. Inner automorphisms from $N$ act trivially there, while $H$ acts faithfully on $V$, so $h = I$. Then $x$ belongs to $Z(N) = Z$. Finally $s = -I$ acts as $-1$ on $Z$, so it fixes only 0. Consequently $Z(G) = 1$.

## 4. Centralizer bound sufficient by itself to falsify the question

$A = W + Z$ is an abelian normal subgroup of $G$ of order $5^{52}$, and $Z$ is normal of order $5^{16}$. The whole group $G$ acts trivially on $A/Z$: $H$ fixes $W$, and all brackets $[L, A]$ lie in $Z$. For any $g$ in $G$ the map

$$A \longrightarrow Z, \qquad \alpha \longmapsto \alpha^{-1} g^{-1} \alpha g$$

is a homomorphism, because $A$ is abelian and normalized by $g$. Its kernel is $C_A(g)$. Thus

$$|C_G(g)| \geq |C_A(g)| \geq |A|/|Z| = 5^{36}$$

for every $g$. If $n$ denotes the actual largest conjugacy class, then

$$n \leq |G|/5^{36} = 120 \cdot 5^{32}, \qquad \frac{|G|}{(120 \cdot 5^{32})^2} = \frac{5^4}{120} = \frac{125}{24} > 1.$$

Already this proves $|G| > n^2$. No estimate about an unexamined family or an incomplete enumeration of conjugacy classes is needed.

## 5. Exact largest conjugacy class

We now show $\min_g |C_G(g)| = 4 \cdot 5^{36}$, strengthening the preceding bound. Write $g = (x, h)$.

If $h - I$ is invertible on $U$, it is invertible on $V$ and $Z$. Conjugation by a vector $v$ in $V$ changes the $V$ component of $x$ to $v + (I - h)v$. Choose $v$ to make this zero. The $N$ component now lies in $A$. Conjugating by a suitable element of $Z$ then kills its $Z$ component. Thus $g$ is conjugate to $(w, h)$ for some $w$ in $W$.

An element of $N$ centralizing $(w, h)$ has zero image in $V$, so lies in $A$. Within $A$ the $W$ factor $w$ has trivial conjugation action, and the fixed space of $h$ is exactly $W$. Therefore $C_N((w, h)) = W$. Every element of $C_H(h)$ centralizes $w$, since $H$ fixes $W$ pointwise. Projection to $H$ therefore gives

$$|C_G((w, h))| = 5^{36} |C_H(h)|.$$

Every centralizer in $\mathrm{SL}_2(\mathbb{F}_5)$ has order at least 4. One quick proof: if an element has order at least 4, use its cyclic subgroup; the identity has centralizer $H$; the only involution is $-I$ and is central; an element of order 3 commutes with $-I$ and generates with it a subgroup of order 6. The assertion about involutions follows from the distinct roots of $X^2 - 1$ and the determinant-one condition.

If $h - I$ is singular and $h$ is not $I$, its characteristic polynomial is $(X - 1)^2$. Thus $h = I + J$, $J^2 = 0$, $J$ nonzero, and $h$ has order 5. The image of $C_G(g)$ in $H$ contains $\langle h \rangle$, while its kernel contains $C_A(g)$. Hence $|C_G(g)| \geq 5 \cdot 5^{36}$.

It remains to consider $h = I$, so $g$ lies in $N$. If $g$ lies in $A$, its centralizer contains all of $A$, of order $5^{52}$. Otherwise the image of $C_N(g)$ in $N/A$ contains the nonzero element $gA$, which has order 5, and its kernel is $C_A(g)$, of order at least $5^{36}$. Again its centralizer has order at least $5 \cdot 5^{36}$.

Finally take $h_0 = \mathrm{diag}(2, 3)$. It has distinct eigenvalues different from 1, and its centralizer in $\mathrm{SL}_2(\mathbb{F}_5)$ consists of the four matrices $\mathrm{diag}(t, t^{-1})$, $t \in \mathbb{F}_5^*$. The element $(0, h_0)$ consequently has centralizer of order $4 \cdot 5^{36}$. This proves the asserted minimum and

$$n = \frac{120 \cdot 5^{68}}{4 \cdot 5^{36}} = 30 \cdot 5^{32}.$$

The exact failed inequality is

$$\frac{|G|}{n^2} = \frac{120 \cdot 5^{68}}{900 \cdot 5^{64}} = \frac{250}{3} > 1.$$

## 6. Replay and scope of the computation

The source gives the following command, run from its original workspace root (historical command, preserved here as a record):

```
python tools/krun.py -name replay-2030 -timeout 180 -
python witness_2030.py
```

The construction and proof above do not depend on software. The script uses only the Python 3 standard library. Sparse vector entries and Gaussian elimination are reduced modulo 5; cardinalities use arbitrary-precision Python integers. The run was checked using Python 3.12.14.

The script first checks the BCH associator as a polynomial in the free anticommutative algebra truncated at degree three, obtaining exactly −Jacobi over $\mathbb{F}_5$. It also checks the formal identity $x * y - y * x = [x, y]$ and the inverse formula. The script then reconstructs all 68 basis vectors, checks all 314432 basis Jacobi identities and all 4624 antisymmetry identities, computes the lower central dimensions $68, 52, 16, 0$ and centre dimension 16, checks the rank-36 map $W \longrightarrow \mathrm{End}(E)$, enumerates all 120 matrices in $H$ and their derived subgroup, and checks preservation of all basis brackets by generators of $H$. It checks that $A$ is abelian, $[L, A]$ is contained in $Z$, $H$ fixes $W$, and $H$ has no nonzero fixed vector on $Z$. It also checks the complement centralizer and fixed space for the attaining element. The finite output is `data/witness_certificate.json`.

No enumeration of the $120 \cdot 5^{68}$ group elements is asserted. The all-element centralizer arguments in Sections 4 and 5 are mathematical certificates. The final arithmetic checks in the script implement their consequences.

## Provenance and verification status

Sections 1–6 reproduce `construction_2030.txt` from the imported run

`counterexample-only-astra-all-20260906-01-20.30-31a75117fb-e0-r0`.

Its task record names model `gpt-6-astra`; its solver record reports `COMPLETE_COUNTEREXAMPLE` and its validator record reports `PASS`. Those are historical records, not repository verifier verdicts. The source supplies its own arguments for the elementary group and Lie algebra facts and lists no external theorem citation. No new theorem, citation, or claim of novelty has been added in this transcription. Scoped symbols were renamed only to avoid collisions; the complete map is in `verification/import-packaging.md` under the problem folder.

This draft is stored under `attempts/`, not `solution/`. The original proof, script, task and result records, and certificates are preserved byte for byte under `experiments/imported-counterexample/`; their SHA-256 manifest records the source and copy hashes. Local packaging and replay records belong to that import package. Independent repository mathematical verification, citation and novelty review, and GAP corroboration remain pending. A Python replay is not independent GAP corroboration and does not establish novelty.