# A COUNTEREXAMPLE TO KOUROVKA 20.86

Abstract. We give a finite group of order 719,323,136 for which the inequality in Kourovka Problem 20.86 fails at $p = 7$. The proof gives the Sylow number and the number of 7-elements directly. This is an unverified draft awaiting verification.

## 1. Problem and result

Kourovka Problem 20.86, proposed by P. Schmid, asks whether

$$\ell^p \geq k^{p-1}$$

always holds for the following parameters. Let $G$ be a finite group and $p$ a prime, let $P$ be a Sylow $p$-subgroup, and let $G_p$ be the set of all $p$-elements of $G$, including the identity. Write

$$s_p(G) = 1 + kp > 1, \qquad f_p(G) = \frac{|G_p|}{|P|} = 1 + \ell(p-1),$$

where $s_p(G)$ is the number of Sylow $p$-subgroups. The target question is transcribed from the *Kourovka Notebook*, arXiv:1401.0300v45, Problem 20.86. Its preliminary printed upper bound for $f_p(G)$ is omitted: as recorded in the source statement, that bound is inconsistent already for $S_3$ at $p = 2$, where $f = 2$ and $k = 1$. Neither that bound nor the distinct contextual inequality $f_p(G)^p \geq s_p(G)^{p-1}$ is used here.

**Theorem 1.** *There is a group $G$ of order 719,323,136 whose Sylow 7-subgroups have order 343, with*

$$s_7(G) = 2{,}097{,}152, \quad f_7(G) = 293{,}479, \quad k = 299{,}593, \quad \ell = 48{,}913.$$

*For this group, $\ell^7 < k^6$. Thus the answer to Problem 20.86 is negative.*

The construction is the standard Heisenberg representation. The proof uses elementary finite fields, linear algebra, semidirect products and Sylow conjugacy; no specialized external theorem is load-bearing. The explicit parameter comparison is the proposed answer to the Notebook question.

## 2. Construction

Let $F = \mathbb{F}_2[z]/(z^3 + z + 1)$, and write $z$ for the residue class. The polynomial has no root in $\mathbb{F}_2$, so $F$ is a field of order 8. Its element $z$ has multiplicative order 7. Let $V = F^7$, viewed additively, with basis $e_j$ indexed by $j \in \mathbb{Z}/7\mathbb{Z}$. Define $F$-linear maps

$$Xe_j = e_{j+1}, \qquad De_j = z^j e_j.$$

Then $DX = zXD$ and $X^7 = D^7 = I$. Consequently

$$P = \langle X, D \rangle = \{z^c X^b D^a : a, b, c \in \mathbb{Z}/7\mathbb{Z}\}$$

is a group. Its displayed 343 elements are distinct: the permutation of the basis determines $b$, and the seven nonzero entries then determine $a$ and $c$. Since $zI = DXD^{-1}X^{-1}$, every displayed element belongs to $\langle X, D \rangle$. In particular, $|P| = 7^3$.

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Set $G = V \rtimes P$ for this action. Thus $|G| = 8^7 \cdot 7^3 = 719{,}323{,}136$ and $P$ is a Sylow 7-subgroup. We will show that its parameters are

$$s = 2{,}097{,}152, \qquad f = 293{,}479, \qquad k = 299{,}593, \qquad \ell = 48{,}913.$$

## 3. Sylow number

Since $zI \in P$ fixes only 0 in $V$, $C_V(P) = 0$. An element $v$ of $V$ normalizes the embedded complement $P$ precisely when $v$ is fixed by $P$: conjugation by $v$ changes the $V$-coordinate of $(0, u)$ by $(I - u)v$. Thus $N_G(P) = P$ and $s = |G : P| = 8^7 = 2{,}097{,}152$.

## 4. Fixed spaces and 7-elements

The six nonidentity scalar elements $z^c I$ fix only 0. For $u = z^c X^b D^a$ with $(a, b) \neq (0, 0)$, its fixed space has dimension 1. If $b = 0$, its diagonal entries $z^{c+aj}$ run through all seventh roots of unity, so exactly one is 1. If $b \neq 0$, its underlying permutation is a single 7-cycle and the product of its seven weights is $z^{7c+a(0+1+\cdots+6)} = 1$. The equation $uv = v$ then determines all seven coordinates from one freely chosen coordinate. The same description shows $u^7 = I$. Hence every nonidentity element of $P$ has order 7, and there are 336 elements in this second case.

For $u \neq I$ in $P$, an element $(v, u)$ of $G$ has 7-power order exactly when

$$(I + u + \cdots + u^6)v = 0.$$

Indeed its seventh power has that $V$-coordinate; if $(v, u)$ has 7-power order, this seventh power belongs to both the 2-group $V$ and a 7-group, so is trivial. Conversely, this equation gives order 7. In characteristic 2, the linear operator $I + u + \cdots + u^6$ is the projection onto $C_V(u)$: it is the identity there, and its kernel equals $(I - u)V$. Thus the number of 7-elements above $u$ is $8^{7-\dim C_V(u)}$. Above $I$, only the identity of $V$ is a 7-element.

It follows that, including the identity,

$$\begin{aligned} |G_7| &= 1 + 6 \cdot 8^7 + 336 \cdot 8^6 = 100{,}663{,}297, \\ f &= |G_7|/|P| = 293{,}479, \\ k &= (s-1)/7 = 299{,}593, \\ \ell &= (f-1)/6 = 48{,}913. \end{aligned}$$

Finally,

$$\begin{aligned} \ell^7 &= 669{,}838{,}495{,}472{,}739{,}697{,}494{,}750{,}319{,}599{,}217 \\ &< k^6 = 723{,}086{,}029{,}984{,}186{,}130{,}422{,}176{,}990{,}254{,}449. \end{aligned}$$

This is the strict reverse of the Notebook inequality, so its answer is negative.

## Scope

The example answers the stated $\ell, k$ question. No claim about the smallest possible counterexample is made.

## Methods and AI disclosure

The user has classified this campaign result as **unverified**. The automated checks described here are historical records; verification remains pending.

This note was produced by an automated research pipeline using the large language model `gpt-6-astra` through Codex. The candidate passed two independent blind adversarial verification passes and a citation audit. Its finite construction and arithmetic were independently reproduced in Python and GAP, covering all 343 complement elements; the reviewers also checked all 117,649 ordered products in independent Python programs, and one used a distinct binary matrix representation in GAP. The proof does not infer the ambient group's counts from enumeration. Scripts, transcripts and reports are preserved under `problems/20.86/` in the accompanying repository, including `verification/r1c2-v1.md`, `verification/r1c2-v2.md` and `experiments/README.md`. The proof text is faithfully converted from the recorded `solution/solution.md`. The bounded novelty search found no exact prior answer; it does not certify originality. This is an unverified draft awaiting verification, with no claim of Notebook acceptance.