A NEGATIVE ANSWER TO KOUROVKA 21.113(A)

Abstract. For a finite group G and prime p, $\Psi_{p,G}$ is zero on p-singular elements; at a p-regular element x it counts the p-elements of $C_G(x)$, including 1. Problem 21.113(a) asks whether this is always an ordinary character.

The explicit group below has order 58320. At $p = 2$ an irreducible character χ of degree 18 has $\langle \Psi_{2,G}, \chi \rangle = -1$. Thus the answer to (a) is negative. This is an unverified draft awaiting verification. The conditional projectivity question in (b) for other pairs is not settled.

1. The problem

Problem 21.113 of the *Kourovka Notebook*, as recorded in `1401.0300v45.pdf`, reads as follows.

> **21.113.** Let G be a finite group and p be a prime. Let $\Psi_{p,G}$ be the class function of G which vanishes on all p-singular elements of G and whose value at each p-regular element x of G is the number of p-elements of $C_G(x)$.
>
> (a) Is it true that $\Psi_{p,G}$ is a character of G?
>
> (b) If yes, can $\Psi_{p,G}$ be afforded by a projective RG-module, where R is a complete discrete valuation ring of characteristic zero such that the field of fractions of R is a splitting field for G and its subgroups, and the residue field $R/J(R)$ is a splitting field of characteristic p for G and its subgroups?
>
> It is known that $\Psi_{p,G}$ is a character when $G \cong S_n$ for any positive integer n and any prime p (T. Scharf, *J. Algebra*, 139, no. 2 (1991), 446–457).
>
> G. Robinson

No external literature theorem is a load-bearing premise. The averaging argument is proved below. The new construction selects the stated matrix embedding and the negative trace pattern; finite premises have three independent computational realizations.

2. Exact group

Work over $\mathbb{F}_3$, with row vectors $V = \mathbb{F}_3^4$ and

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Define $T = \langle a, b \rangle$, with the following exact generators:

$$a = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 \end{pmatrix}.$$

These matrices satisfy $aJa^\top = bJb^\top = J$, $uJu^\top = -J$, $u^2 = -I$, and $ua = au$, $ub = bu$. Set $H = \langle T, u \rangle$.

Let $E = V \times \mathbb{F}_3$, with

$$(v, z)(w, t) = (v + w, z + t + 2vJw^\top).$$

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For $h \in H$, write $hJh^\top = \mu(h)J$. Its action on E is $(v,z)^h = (vh, \mu(h)z)$. Form the split group $G = E : H$ and $K = E : T$. The center $Z = 0 \times \mathbb{F}_3$ of E is fixed by T and inverted by u.

The following finite data are certified by the supplied GAP script:

$$|T| = 120, \quad -I \in T, \quad |H| = 240, \quad |E| = 243, \quad |K| = 29160, \quad |G| = 58320.$$

The group T is perfect; identifying it with $\mathrm{SL}_2(5)$ is descriptive and is not a premise. Its classes have (order, size)

$$(1,1),\ (2,1),\ (3,20),\ (4,30),$$
$$(5,12),\ (5,12),\ (6,20),\ (10,12),\ (10,12).$$

The two classes of elements of order 5 are denoted C_5 and C_5'. There is an irreducible character ϕ of K of degree 9, nontrivial on Z, whose values on the complement's odd-order elements are

$$\phi(1) = 9, \qquad \phi(C_3) = -3, \qquad \phi(C_5) = \phi(C_5') = -1. \tag{A}$$

3. Odd-part averaging identity

For a finite group L and an irreducible character η, the conjugate of $\langle \Psi_{2,L}, \eta\rangle$ is

$$\frac{1}{|L|}\sum_{g\in L} \eta(g_{2'}). \tag{1}$$

Indeed the fiber of the odd-part map over an odd element x consists exactly of yx, where y is a two-element in $C_L(x)$. The value we compute below is real, so the conjugate convention is immaterial.

For $h \in H$ write $h = rs = sr$, where r is its two-part and s its odd part. For any character η of G,

$$\frac{1}{|E|}\sum_{e\in E} \eta\big((eh)_{2'}\big) = \frac{1}{|C_E(r)|}\sum_{c\in C_E(r)} \eta(cs). \tag{2}$$

Here is a proof specialized to the defined split action. Every two-element in Er is E-conjugate to r. If such an element is $(v,z)r$, its $|r|$-th power is trivial, giving $N_r v = 0$. Since $|r|$ is invertible in $\mathbb{F}_3$, $\ker N_r = \mathrm{im}(r-1)$, so E-conjugation clears its vector coordinate. If $\mu(r) = 1$ the remaining central coordinate vanishes from the same power equation; if $\mu(r) = -1$ it is cleared by central conjugation. Consequently every eh is conjugate under E to rcs for $c \in C_E(r)$. Conversely rcs has two-part r, since c, s generate an odd-order group. The map $(x,c) \mapsto x^{-1}rcsx$ from $E \times C_E(r)$ to Eh is surjective; each fiber has $|C_E(r)|$ elements, because its allowed x form a coset of $C_E(r)$, after which c is determined. Summing the character gives (2).

Take ϕ from (A), and put $\chi = \mathrm{Ind}_K^G(\phi)$. It is irreducible of degree 18: K has index two, and its two conjugate characters have the distinct nontrivial Z-characters, interchanged by u. In a unitary representation D affording ϕ, put

$$\alpha_r(s) = \mathrm{trace}\big(D(s) \text{ on } D^{C_E(r)}\big)$$

for r outside T and $s \in C_T(r)$. For $h \in T$, $C_E(h_2)$ contains Z; the invariant spaces of both constituents of χ vanish. For h outside T, its odd part lies in T. The two constituent sums have equal totals, because outside conjugation permutes the outside coset. Their factor two cancels the index $[H : T]$. Applying (2), then collecting the unique commuting pairs r, s by T-conjugacy classes, gives

$$\langle \Psi_{2,G}, \chi\rangle = \sum_{r\in (H\setminus T)_2/T} \frac{1}{|C_T(r)|} \sum_{\substack{s\in C_T(r)\\ |s| \text{ odd}}} \alpha_r(s). \tag{3}$$

The uses of ϕ in (2) are legitimate: all the evaluated odd parts and the E-conjugations belong to K. Taking conjugates throughout would give the same real value below.

4. Evaluation

There are precisely three T-classes of outside two-elements:

Representative type	Class size	Order	Centralizer order	$\dim V^r$
u	1	4	120	0
$-u$	1	4	120	0
u times an order-four element of T	30	2	4	2

This table is a finite premise to be independently certified. It also follows by multiplying T's two-elements by the central u: its only two-elements have orders $1, 2, 4$, and its thirty order-four elements form one class. The fixed dimensions are checked from the matrices.

For $r = \pm u$, $C_E(r) = 1$, so $\alpha_r = \phi$ on T. Thus these two classes together contribute

$$\frac{2}{120}(9 + 20 \cdot (-3) + 12 \cdot (-1) + 12 \cdot (-1)) = -\frac{5}{4}. \tag{4}$$

For the remaining class, $C_E(r)$ is an elementary abelian group of order 9 disjoint from Z. The degree-nine E-irreducible with the chosen nontrivial central character has value zero off Z and degree nine. This follows directly from the Schrödinger representation of E on functions $\mathbb{F}_3^2 \to \mathbb{C}$: translations and phase multipliers have zero trace unless their vector coordinate is zero; their 81 operators are orthogonal and span the full matrix algebra, proving irreducibility. The central-character summand of $\mathbb{C}[E]$ has dimension 81, proving uniqueness. Since ϕ has degree nine and that central character, its restriction is this E-irreducible. Averaging over $C_E(r)$ therefore gives $\alpha_r(1) = 9/9 = 1$. Its complement centralizer has order four, so only the odd identity contributes, giving $1/4$ in (3).

Combining with (4) yields

$$\langle \Psi_{2,G}, \chi \rangle = -\frac{5}{4} + \frac{1}{4} = -1.$$

Hence $\Psi_{2,G}$ is not an ordinary character. This refutes the universal assertion in 21.113(a); for this G it also cannot be the character of a projective module as in (b). No conclusion is asserted about the conditional projectivity question for pairs where Ψ is an ordinary character.

Methods and AI disclosure

The user has classified this campaign result as **unverified**. The automated checks described here are historical records; verification remains pending.

This note was produced by an automated research pipeline built on a large language model (`gpt-6-astra`) using Codex. Before assembly, the candidate passed two independent blind adversarial verification passes and a citation audit; no external literature theorem is a load-bearing premise. The finite data were independently realized in permutation GAP, exact Python over $\mathbb{Q}(\zeta_3)$, and presentation/pc GAP. The completed checks give degree 18, character norm 1, $\Psi_{2,G}(1) = 3808$ and coefficient -1. Scripts and completed transcripts are preserved in `problems/21.113/experiments/r1-counterexample/`; the two reports and citation audit are in `problems/21.113/verification/`, under `r1c4-v1.md`, `r1c4-v2.md`, and `r1c4-citation-fit.md`. The construction and negative trace pattern are the new argument presented here. This draft remains unverified and is a candidate awaiting human review; no claim of acceptance by the Notebook is made.