A sharp bound for complements of finite coset unions

An affirmative answer to Kourovka Notebook Problem 21.115

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Abstract

Let $C_1, \ldots, C_n$ be arbitrary left or right cosets of possibly different subgroups of a finite group G, and suppose that their union is proper. We prove the sharp estimate

$$\left| G \setminus \bigcup_{i=1}^{n} C_i \right| \geq \frac{|G|}{2^n},$$

thereby answering Kourovka Notebook Problem 21.115 affirmatively. For each coset we construct a matrix of rank at most two whose zero pattern records membership of quotients $x^{-1}y$ in that coset. The entrywise product of these matrices has rank at most 2^n, while its nonzero diagonal and its row supports force rank at least $|G|/|A|$, where A is the complement. Equality occurs for the coordinate index-two cosets in $K \times (C_2)^n$.

1 Statement

Problem 21.115 asks whether, for left or right cosets $C_1, \ldots, C_n$ of a finite group G whose union U is not all of G, one always has $|G \setminus U| \geq |G|/2^n$ [1].

Theorem 1. *Let G be a finite group and let $C_1, \ldots, C_n$ be left or right cosets of arbitrary subgroups of G. If*

$$U = C_1 \cup \cdots \cup C_n \neq G,$$

then

$$\boxed{|G \setminus U| \geq \frac{|G|}{2^n}.}$$

Equivalently, $|G \setminus U| \geq \lceil |G|/2^n \rceil$.

The proof uses only the following elementary rank facts. For a row vector r, write $\operatorname{supp}(r)$ for the set of its nonzero coordinates. The symbol $\circ$ denotes the Hadamard, or entrywise, product of matrices.

Lemma 2 (Sparse diagonal)**.** *Let M be an $N \times N$ matrix over a field. Suppose every diagonal entry of M is nonzero and every row has at most m nonzero entries. Then*

$$\operatorname{rank} M \geq \frac{N}{m}.$$

Proof. Put $r = \operatorname{rank} M$ and choose r rows which form a basis of the row space. The union of their supports has size at most rm. This union must contain every column index: if column j vanished on all basis rows, it would vanish on every row of M, contrary to $M(j,j) \neq 0$. Hence $N \leq rm$. $\square$

Lemma 3 (Schur-product rank)**.** *For matrices $M_1, \dots, M_n$ of the same size over a field,*

$$\operatorname{rank}(M_1 \circ \cdots \circ M_n) \le \prod_{i=1}^{n} \operatorname{rank} M_i.$$

Proof. Write each M_i as a sum of $r_i = \operatorname{rank} M_i$ rank-one matrices, $M_i = \sum_{j=1}^{r_i} u_{ij} v_{ij}^{\mathsf{T}}$. Distributing the Hadamard product expresses $M_1 \circ \cdots \circ M_n$ as a sum of $\prod_i r_i$ rank-one matrices, since

$$(uv^{\mathsf{T}}) \circ (st^{\mathsf{T}}) = (u \circ s)(v \circ t)^{\mathsf{T}}.$$

□

2 Proof of the theorem

The case $n = 0$ is immediate, so assume $n \ge 1$. Set

$$A = G \setminus U, \qquad N = |G|, \qquad m = |A|.$$

Thus $m > 0$. Every right coset Ha may be written as the left coset

$$Ha = a(a^{-1}Ha).$$

We may therefore write every C_i as $a_i H_i$, with $H_i \le G$. Choose $t \in A$ and left-translate the entire configuration by t^{-1}. This preserves all cardinalities and leaves every C_i a left coset, so we may assume from now on that

$$1 \in A. \tag{1}$$

For each i, choose an injective labelling

$$\lambda_i : G/H_i \longrightarrow \mathbb{Q}$$

of the finite set of left cosets of H_i. Define an $N \times N$ matrix, with rows and columns indexed by G, by

$$M_i(x, y) = \lambda_i(yH_i) - \lambda_i(xa_iH_i) \qquad (x, y \in G).$$

The first term depends only on y and the second only on x; hence $\operatorname{rank} M_i \le 2$. More importantly, injectivity of λ_i gives the exact zero condition

$$\begin{aligned} M_i(x, y) = 0 &\iff yH_i = xa_iH_i \\ &\iff x^{-1}y \in a_iH_i = C_i. \end{aligned} \tag{2}$$

Now form the Hadamard product

$$M = M_1 \circ \cdots \circ M_n.$$

By Lemma 3,

$$\operatorname{rank} M \le 2^n. \tag{3}$$

On the other hand, because the coefficient field has no zero divisors, (2) implies

$$\begin{aligned} M(x, y) \ne 0 &\iff x^{-1}y \notin C_i \quad \text{for every } i \\ &\iff x^{-1}y \in A. \end{aligned}$$

Consequently the support of row x is exactly

$$\{y \in G : x^{-1}y \in A\} = xA,$$

and therefore every row has exactly m nonzero entries. Moreover, (1) shows that $M(x, x) \neq 0$ for every $x \in G$. Lemma 2 now yields

$$\operatorname{rank} M \geq \frac{N}{m}. \tag{4}$$

Combining (3) and (4),

$$\frac{|G|}{|A|} = \frac{N}{m} \leq 2^n,$$

which is precisely $|A| \geq |G|/2^n$. This proves Theorem 1. □

3 Sharpness

The constant 2^{-n} cannot be improved. Let K be any finite group and put

$$G = K \times (C_2)^n.$$

Write elements of $(C_2)^n$ as binary vectors $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. For $1 \leq i \leq n$, let

$$C_i = \{(k, \varepsilon) \in G : \varepsilon_i = 1\}.$$

This is a coset of the index-two subgroup defined by $\varepsilon_i = 0$. The union consists of the elements whose binary coordinate vector is nonzero, and hence

$$G \setminus \bigcup_{i=1}^{n} C_i = K \times \{0\}^n, \qquad \left| G \setminus \bigcup_{i=1}^{n} C_i \right| = |K| = \frac{|G|}{2^n}.$$

Thus Theorem 1 is best possible for every n.

Relation to earlier work and AI-assistance statement

Sambale and Tărnăuceanu proved the general lower bound $|G|/(2n!)$ and established the sharp $|G|/2^n$ bound in several special cases, including elementary abelian groups; they formulated the general bound proved here as a conjecture [2]. A public online manuscript using essentially the same matrix-rank method is also available [3]. No claim of priority is made in this note.

This is a ChatGPT-assisted proof. ChatGPT (OpenAI) was used substantially in finding and refining the rank construction, checking the argument, and preparing the exposition. The named author is responsible for checking the final manuscript and for all mathematical claims it contains.

References

[1] E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved Problems in Group Theory: The Kourovka Notebook*, No. 21, 2026, Problem 21.115, arXiv:1401.0300, `https://arxiv.org/abs/1401.0300`.

[2] B. Sambale and M. Tărnăuceanu, *On the size of coset unions*, J. Algebraic Combin. **55** (2022), 979–987, doi:10.1007/s10801-021-01079-x.

[3] `DannyExperiments`, *A Rank Method for Complements of Finite Coset Unions*, version 1.0.1, 2026, doi:10.5281/zenodo.21762471.