RESIDUALLY FINITE GOLOD GROUPS WITH CENTRE OF PRIME ORDER

Abstract. For every prime p we construct an infinite three-generated residually finite Golod p-group with centre C_p. Starting with Golod's graded nil algebra, a countable homogeneous central quotient produces a centreless algebra without destroying its degree filtration. A square-zero direct summand then supplies the prescribed finite centre.

1. The problem and the construction

Problem 21.132 of the Kourovka Notebook [2] asks:

> Based on the development of E. S. Golod's construction (see, for example, Discrete Math. Appl., 23, no. 5–6 (2013), 491–501), for each prime number p, construct a finitely generated residually finite p-group with a non-trivial finite centre. Such groups with infinite and trivial centres are known (see 9.76 and Archive 11.101).
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We construct an infinite group, as intended by the surrounding references. We also retain the precise Golod-group condition in Notebook 9.76: the group is generated by the elements $1+x_i+I$ in the adjoint group of a nil, nonnilpotent quotient of a free nonunital associative algebra on at least two specified generators over a field of characteristic p.

Theorem 1. *For every prime p there is an infinite three-generated residually finite Golod p-group G with $Z(G) \cong C_p$.*

Golod's graded nil-algebra construction [1, Example 1] is the only external existence theorem used. The idea of removing central nil-algebra elements occurs in Sereda–Sozutov [3, proof of Theorem 1]. Here an explicit homogeneous iteration preserves the separating degree filtration. Residual finiteness is proved from that filtration; no residual-finiteness assertion about an arbitrary quotient is imported. Adjoining one square-zero algebra generator then gives the required nontrivial finite centre.

All algebras below are associative and nonunital over $\mathbb{F}_p$. For an algebra R, write $R^{\#}$ for its unitization and

$$Z_{\mathrm{alg}}(R) = \{a \in R : ab = ba \text{ for all } b \in R\}.$$

An algebra is *nil* when every element is nilpotent and *nilpotent* when $R^N = 0$ for some positive integer N. Parentheses around a subset denote its generated two-sided ideal.

2. A centreless graded quotient

Lemma 2. *If R is a finitely generated nonnilpotent nil algebra, then $R/(Z_{\mathrm{alg}}(R))$ is not nilpotent.*

Proof. Suppose $R^N \subseteq (Z_{\mathrm{alg}}(R))$. The finitely many words of length N in a finite algebra generating set generate R^N as an ideal. Their expressions in $(Z_{\mathrm{alg}}(R))$ involve finitely many central elements $c_1, \dots, c_t$. Hence $R^N \subseteq K = (c_1, \dots, c_t)$. Choose positive integers e_i with $c_i^{e_i} = 0$ and put $e = 1 + \sum_i (e_i - 1)$. Centrality implies $K^e = 0$: each term in a product of e generators of this ideal contains some $c_i^{e_i}$. Thus $R^{Ne} = 0$, a contradiction. If no nonzero central element was needed in the expressions, already $R^N = 0$. □

Date: Draft: September 15, 2026.

Golod [1, Example 1, pp. 274–275] constructs a two-generated positively graded nil, nonnilpotent algebra

$$A = \mathbb{F}_p\langle x_1, x_2\rangle_+/I = \bigoplus_{j\geq 1} A_j,$$

where I is homogeneous, $\dim_{\mathbb{F}_p} A_j < \infty$, and the images of x_1, x_2 have degree one. Here the subscript $+$ means the positive-degree part. We use the same symbols for the generator images in A. The homogeneous relations admit an effective enumeration: in Golod's induction the coefficient forms in sufficiently high powers of generic polynomials are taken, with exponents satisfying explicit bounds.

Define ideals by

$$J_0 = 0, \qquad J_{n+1}/J_n = (Z_{\mathrm{alg}}(A/J_n)), \qquad J = \bigcup_{n\geq 0} J_n, \qquad Q = A/J.$$

These ideals are homogeneous. Indeed, in an algebra generated in degree one, commuting a central element with each generator shows separately that all its homogeneous components are central. The ideal generated by the centre is therefore homogeneous. Every A/J_n is nil and finitely generated; Lemma 2 shows inductively that it is not nilpotent.

The algebra Q is not nilpotent. If $Q^N = 0$, every length-N word in x_1, x_2 lies in J. There are finitely many such words, so they all lie in one J_n. Since these words generate A^N as an ideal, this gives $(A/J_n)^N = 0$, a contradiction. In particular Q is infinite dimensional: a finite-dimensional positively graded algebra is zero in sufficiently high degrees and hence nilpotent.

Moreover $Z_{\mathrm{alg}}(Q) = 0$. If $a + J$ is central, both commutators $[a, x_i] = ax_i - x_ia$ lie in J, hence in some common J_n. Then $a + J_n$ is central in A/J_n, so $a \in J_{n+1}$ and $a + J = 0$. Thus the countable iteration has already killed the whole algebra centre.

The construction also gives recursively enumerated homogeneous relations over $\mathbb{F}_p$. Start with Golod's effective relation list. For each homogeneous free polynomial f, enumerate finite ideal-membership certificates for $[f, x_1]$ and $[f, x_2]$, dovetailing all searches. When both have certificates using relations already listed, add f. All certificates are finite polynomial identities. Each addition therefore kills a central nilpotent element of the current quotient. Fair dovetailing enumerates the least ideal closed under this rule, which is the preimage of J: the successive J_n are homogeneous, every certificate uses finitely many relations, and every rule consequence appears at a finite stage. This requires no decision procedure for ideal membership.

3. The group and its finite centre

Proof of Theorem 1. Let q_1, q_2 be the images of x_1, x_2 in Q. Form the algebra and group

$$B = Q \oplus \mathbb{F}_p z, \qquad z^2 = Qz = zQ = 0, \qquad G = \langle 1+q_1, 1+q_2, 1+z\rangle \leq 1 + B.$$

The algebra B is generated by q_1, q_2, z. It is nil, since $(q + \lambda z)^\ell = q^\ell$ for $q \in Q$, $\lambda \in \mathbb{F}_p$ and $\ell \geq 2$, and is not nilpotent because it maps onto Q. It is a quotient of the free nonunital algebra on the three displayed generators. Thus G is a Golod group in the stated sense.

Every $1 + b$, $b \in B$, is invertible by a finite geometric series. Moreover

$$(1+b)^{p^s} = 1 + b^{p^s} = 1$$

once p^s is at least the nilpotence index of b. Hence G is a p-group, for every prime p, including $p = 2$. Its $\mathbb{F}_p$-linear span in $B^\#$ is a unital subalgebra, since products of group elements remain in G. This span contains $q_1 = (1 + q_1) - 1$, q_2 and z, and therefore equals $B^\#$. If G were finite, $B^\#$ would be finite dimensional, contrary to the infinite dimension of Q. Thus G is infinite.

Give z degree one and retain the grading of Q. For $m \geq 2$ the ideal

$$B_{\geq m} = \bigoplus_{j\geq m} B_j$$

has finite codimension, and $\bigcap_{m\geq 2} B_{\geq m} = 0$. The maps

$$G \longrightarrow 1 + (B/B_{\geq m})$$

are homomorphisms into finite groups of order $p^{\dim_{\mathbb{F}_p}(B/B_{\geq m})}$. Indeed, their targets have underlying set $\{1 + b : b \in B/B_{\geq m}\}$ and adjoint multiplication $(1 + b)(1 + b') = 1 + b + b' + bb'$. Every nonidentity $1 + b \in G$ survives once m exceeds the largest degree occurring in b. This proves residual finiteness, with finite p-group quotients.

If $1 + b \in Z(G)$, it commutes with the three generators of G. Therefore b commutes with q_1, q_2, z and with all of B. Writing $b = q + \lambda z$ gives $q \in Z_{\mathrm{alg}}(Q) = 0$. Conversely every $1 + \lambda z$ is central and equals $(1 + z)^{a_\lambda}$, where $0 \leq a_\lambda < p$ is the integer representing $\lambda \in \mathbb{F}_p$. Consequently

$$Z(G) = \{1 + \lambda z : \lambda \in \mathbb{F}_p\} \cong C_p.$$

□

4. Further questions

The finite centre is supplied by a direct factor: the construction has $G = \langle 1+q_1, 1+q_2\rangle \times \langle 1+z\rangle$. It does not address the additional requirement of direct indecomposability or a two-generator construction.

Methods and AI disclosure

This internal candidate was produced with an automated research pipeline using `gpt-6-astra` through Codex. Before assembly, two independent adversarial reviews checked the complete proof and a separate source audit checked Golod's construction and the method attribution. The homogeneous central quotient, its effective relation enumeration, residual finiteness and exact centre calculation are proved here. No computation is a premise of the proof. A bounded literature search found no explicit prior full solution and does not establish priority. Human review remains pending. Source notes and verification records are retained in the accompanying repository under `problems/21.132/`.

References

1. E. S. Golod, *On nil-algebras and finitely approximable p-groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), no. 2, 273–276, In Russian.
2. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory. The Kourovka Notebook. No. 21*, Sobolev Institute of Mathematics, Novosibirsk, 2026, arXiv:1401.0300v45, 3 July 2026.
3. V. A. Sereda and A. I. Sozutov, *Associative nil-algebras and Golod groups*, Algebra i Logika **45** (2006), no. 2, 231–238, In Russian; English translation in Algebra and Logic 45 (2006), 134–138.