A Counterexample to Problem 21.148 in the Kourovka Notebook

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Abstract

Let W be the group of increasing bijections of $\mathbb{Q}$ with bounded displacement. We prove that its right-relatively convex subgroups are exactly its isolated subgroups and exactly its convex ℓ-subgroups. Their lattice is therefore distributive but not a chain, giving a negative answer to Kourovka Notebook Problem 21.148. The main ingredient is an orbital sandwich theorem, proved using a fixed eight-factor decomposition and a four-map coding construction adapted from Hyde, Jonušas, Mitchell and Péresse. We give a direct proof of the coding identities using a sign-reversing translation by 4, with a uniform displacement bound.

1 The problem and the counterexample

Question. The following problem appears in the Kourovka Notebook [5, p. 189].

> **21.148.** (V. M. Kopytov, N. Ya. Medvedev). Is it true that the lattice of right-relatively convex subgroups of a right-orderable group is distributive if and only if it is a chain?
>
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Write $\mathcal{R}(G)$ for the lattice of right-relatively convex subgroups, ordered by inclusion. The counterexample below is a right-orderable group for which this lattice is distributive but is not a chain.

For $g \in \mathrm{Aut}(\mathbb{Q}, <)$ put

$$\|g - \mathrm{id}\|_\infty = \sup_{q\in\mathbb{Q}} |qg - q|,$$

and define

$$W = \{g \in \mathrm{Aut}(\mathbb{Q}, <) : \|g - \mathrm{id}\|_\infty < \infty\}. \tag{1}$$

Theorem 1.1 (Main theorem)**.** *The group W is right-orderable and*

$$\mathcal{R}(W) = \mathrm{Isol}(W) = \mathrm{Conv}_\ell(W),$$

where $\mathrm{Isol}(W)$ *is the collection of isolated (root-closed) subgroups and* $\mathrm{Conv}_\ell(W)$ *the collection of convex ℓ-subgroups for the pointwise lattice order. Consequently, $\mathcal{R}(W)$ is distributive but is not a chain. In particular, Problem 21.148 has a negative answer.*

The key step is to force lattice convexity from root-closure. We first factor a bounded automorphism into a fixed finite list of conjugates of sparsely supported maps. A bounded coding construction then expresses each such map in terms of four auxiliary maps that root-closure forces into the subgroup. Uniform displacement bounds allow this construction to be patched simultaneously over all orbitals of a given element. The remaining steps use standard facts about convex ℓ-subgroups.

Throughout, permutations act on the right:

$$q(fg) = (qf)g, \qquad x^y = y^{-1}xy, \qquad [x, y] = x^{-1}y^{-1}xy.$$

For a bijection $\alpha : X \to Y$ and a permutation g of X, the transported permutation g^α of Y is defined by

$$yg^\alpha = ((y\alpha^{-1})g)\alpha \qquad (y \in Y).$$

This agrees with conjugation when $X = Y$. The identity automorphism is denoted by 1.

2 Standard preliminaries

We recall standard facts about ordered permutation groups, relative convexity, and root-closure, fixing the right-action conventions used below. No novelty is claimed for the preliminaries in this section. For the particular set W, we explicitly check that bounded displacement is preserved by the group and lattice operations. The main argument is the bounded coding construction and the orbital sandwich theorem in Sections 3–4.

For $f, g \in \mathrm{Aut}(\mathbb{Q}, <)$ define

$$f \le g \iff qf \le qg \quad \text{for every } q \in \mathbb{Q}.$$

Pointwise minimum and maximum are denoted by $f \wedge g$ and $f \vee g$.

Lemma 2.1. *The set W is a subgroup of* $\mathrm{Aut}(\mathbb{Q}, <)$ *and, with the pointwise order, is an ℓ-group. Moreover, W is right-orderable.*

Proof. The identity belongs to W, since $\|1 - \mathrm{id}\|_\infty = 0$. For $f, g \in W$,

$$|qfg - q| \le |qfg - qf| + |qf - q| \le \|g - \mathrm{id}\|_\infty + \|f - \mathrm{id}\|_\infty.$$

Thus $fg \in W$. Substituting $q = pf$ gives $\|f^{-1} - \mathrm{id}\|_\infty = \|f - \mathrm{id}\|_\infty$, so $f^{-1} \in W$ as well. Hence W is a subgroup of $\mathrm{Aut}(\mathbb{Q}, <)$.

It is standard that $\mathrm{Aut}(\mathbb{Q}, <)$ is an ℓ-group under the pointwise order; see, for example, [4, Lemma 2.2]. The pointwise minimum and maximum of $f, g \in W$ have displacement bounded by $\max\{\|f - \mathrm{id}\|_\infty, \|g - \mathrm{id}\|_\infty\}$. Hence W is an ℓ-subgroup.

Finally, $\mathrm{Aut}(\mathbb{Q}, <)$ is right-orderable by the classical first-difference construction: enumerate $\mathbb{Q}$ and compare the images of the first rational on which two automorphisms differ. With our right-action convention this order is right-invariant, because postcomposition by an increasing bijection preserves the first differing coordinate and its inequality. Restricting this standard order to W proves right-orderability. □

We recall the standard ordered-coset criterion and its intersection consequence; compare [1].

Lemma 2.2 (Standard ordered-coset criterion)**.** *Let G be right-orderable and let $H \le G$. The following are equivalent.*

(i) *H is convex in some right order on G;*

(ii) *the right-coset set $H\backslash G$ admits a total order invariant under the natural right action of G;*

(iii) *H is the stabilizer of a point in a linearly ordered right G-set.*

Arbitrary intersections of right-relatively convex subgroups are right-relatively convex.

Proof. The equivalence of (ii) and (iii) is immediate. If H is convex in a right order on G, its right cosets are convex intervals and inherit a total order preserved by the right action. Conversely, assume that $H\backslash G$ is right G-ordered and choose a right order on the subgroup H. Order G lexicographically: first compare Hg and Hk; when these cosets agree, compare $gk^{-1} \in H$ with 1. The resulting order is right-invariant and makes H convex. This is the right-handed form of the usual relative convexity criterion; compare [1].

For the intersection assertion, represent each H_i as the stabilizer of a point x_i in a linearly ordered right G-set X_i. Well-order the index set and put the lexicographic order on $\prod_i X_i$. The diagonal right action preserves this order, and the stabilizer of $(x_i)_i$ is $\bigcap_i H_i$. □

Definition 2.3. A subgroup $H \leq G$ is *isolated*, or *root-closed*, if

$$x^n \in H \text{ for some } n \geq 1 \quad \Longrightarrow \quad x \in H.$$

Lemma 2.4 (Standard root-closure observation)**.** *Every right-relatively convex subgroup of a right-orderable group is isolated.*

Proof. Let H be convex in a right order and suppose $x^n \in H$. The cases $n = 1$ and $x = 1$ are immediate, so assume $n \geq 2$ and $x \neq 1$. If $1 < x$, then $1 < x < x^2 < \cdots < x^n$; if $x < 1$, then $x^n < \cdots < x^2 < x < 1$. In either case, x lies between two elements of H, so convexity gives $x \in H$. □

The standard inclusion chain. For an arbitrary ℓ-group G with its specified lattice order, let $\mathrm{Conv}_\ell(G)$ denote the family of convex ℓ-subgroups. Relative convexity, in contrast, allows any total right order on the underlying group. The standard inclusions are

$$\boxed{\mathrm{Conv}_\ell(G) \subseteq \mathcal{R}(G) \subseteq \mathrm{Isol}(G).}$$

The first follows from prime-subgroup separation; we recall the general argument in Proposition 5.3. The second is Lemma 2.4. Consequently, the additional inclusion

$$\mathrm{Isol}(G) \subseteq \mathrm{Conv}_\ell(G)$$

implies

$$\mathrm{Conv}_\ell(G) = \mathcal{R}(G) = \mathrm{Isol}(G).$$

For $G = W$, proving this additional inclusion is the special step of the paper: it follows from the orbital sandwich theorem (Theorem 4.2) via Proposition 5.2.

Lemma 2.5 (Square-root forcing)**.** *Let H be an isolated subgroup of a group G. If $t \in H$ and $x^t = x^{-1}$, then $x \in H$. Also, if $y \in H$ and $(xy)^2 = y^2$, then $x \in H$.*

Proof. The relation $x^t = x^{-1}$ also gives $txt^{-1} = x^{-1}$, and hence

$$(xt)^2 = x(txt^{-1})t^2 = t^2 \in H.$$

Thus $xt \in H$ and $x \in H$. For the second assertion, $(xy)^2 = y^2 \in H$ gives $xy \in H$, and therefore $x \in H$. Only closure under square roots is needed in either argument. □

3 Uniform factorization and bounded coding

Let

$$T(q) = q + 1, \qquad B_r = \{g \in \mathrm{Aut}(\mathbb{Q}, <) : \|g - \mathrm{id}\|_\infty \leq r\} \quad (r > 0).$$

For $n \in \mathbb{Z}$ put $J_n = (4n, 4n+2) \cap \mathbb{Q}$, and let

$$S = \left\{ g \in \mathrm{Aut}(\mathbb{Q}, <) : \mathrm{supp}(g) \subseteq \bigcup_{n\in\mathbb{Z}} J_n \right\}. \tag{2}$$

Thus the endpoints of every J_n are fixed by definition. Each $s \in S$ preserves every J_n and is the identity elsewhere, so

$$\|s - \mathrm{id}\|_\infty \leq 2. \tag{3}$$

This is the subgroup denoted by $\mathrm{Stab}(I_4)$ in [4]: an increasing bijection fixing all intervals $(4n+2, 4n+4) \cap \mathbb{Q}$ also fixes their rational endpoints, since it preserves any supremum or infimum that belongs to $\mathbb{Q}$.

Lemma 3.1 (Fixed finite factorization)**.** *Every $w \in B_1$ has a factorization*

$$w = s_1^{T^{r_1}} s_2^{T^{r_2}} \cdots s_8^{T^{r_8}}, \qquad s_j \in S, \tag{4}$$

with the fixed exponent list

$$(r_1, \ldots, r_8) = (0, 2, 1, 3,\ 0, 2, 1, 3).$$

Proof. We use the fragmentation argument of [4, Lemmas 3.2–3.4], with half-unit steps in place of third-unit steps, and include the details.

First, $B_1 \subseteq (B_{1/2})^2$. To see this, let $R(q) = q + 1/2$ and put

$$e = (w^{-1} \vee R^{-1}) \wedge R.$$

Then $e \in B_{1/2}$. We claim that $ew \in B_{1/2}$. If $qw^{-1} \leq q - 1/2$, then $qe = q - 1/2$ and

$$q \leq (q - 1/2)w \leq q + 1/2.$$

If $qw^{-1} \geq q + 1/2$, then $qe = q + 1/2$ and

$$q - 1/2 \leq (q + 1/2)w \leq q.$$

In the remaining case $qew = q$. Thus $w = e^{-1}(ew)$ is the required product of two members of $B_{1/2}$.

Next, let $v \in B_{1/2}$ and put $P = \mathrm{Stab}(2\mathbb{Z})$. Choose an increasing bijection z fixing every odd integer and satisfying

$$(2n)z = (2n)v \qquad (n \in \mathbb{Z}).$$

Such a map exists because $(2n)v \in (2n-1, 2n+1)$: one may interpolate linearly on each of $[2n-1, 2n] \cap \mathbb{Q}$ and $[2n, 2n+1] \cap \mathbb{Q}$. Now $vz^{-1} \in P$ and $z \in \mathrm{Stab}(2\mathbb{Z}+1) = P^T$, so $B_{1/2} \subseteq PP^T$.

Finally, a member of P splits into its restrictions to the two disjoint families of intervals

$$(4n, 4n+2) \cap \mathbb{Q}, \qquad (4n+2, 4n+4) \cap \mathbb{Q}.$$

The first restriction belongs to S, and the second to S^{T^2}. Consequently,

$$B_{1/2} \subseteq PP^T = S\, S^{T^2}\, S^T\, S^{T^3}.$$

Combining this with $B_1 \subseteq (B_{1/2})^2$ gives the stated exponent list. □

Lemma 3.2 (Bounded coding with shift four). *For every $h \in S$ there exist $a, b, c, d \in B_4$ such that*

$$b^{T^4} = b^{-1}, \qquad c^{T^4} = c^{-1}, \qquad d^{T^4} = d^{-1}, \tag{5}$$

$$(ab)^2 = b^2, \tag{6}$$

$$h = [a^{cd}, a^{c^{-1}}]. \tag{7}$$

Proof. A direct construction and verification are given in Appendix A. The maps preserve each interval $(4n-1, 4n+3) \cap \mathbb{Q}$ and fix all its endpoints; therefore their displacements are at most 4. Alternating signs on adjacent intervals produce the three inversion identities, while a second alternating sign inside each interval gives $(ab)^2 = b^2$. Two local commutator factors are then recovered on J_n, and their conjugated supports are disjoint elsewhere. □

Remark 3.3 (Why the shift is four, rather than forty-eight). In the coordinate construction of [4, Section 4], each outer block has width 4, and the coding sites occur every twelve blocks. The residue classes $0, 1, 3, 7$ modulo 12 separate the supports of the four factors packed into their single additional generator and keep the required translated supports separated. A shift of twelve blocks, hence 48 units, interchanges the two sign choices and inverts b, c, d. That packing step is not needed here: all four auxiliary maps may be used separately. We therefore use every block and alternate the sign on consecutive blocks, so translation by 4 suffices. The constants 4 and 8 in our construction are convenient choices, not claimed to be optimal or intrinsic to W.

Choice of scale. The intervals $(4n, 4n+2)$ have length two relative to the unit translation T. Their two alternating families split $\mathrm{Stab}(2\mathbb{Z})$, while translation by T moves this stabilizer to $\mathrm{Stab}(2\mathbb{Z}+1)$. Replacing only J_n by $(2n, 2n+1)$ would not work: the corresponding support subgroup and all its conjugates by integer powers of T fix every integer. They therefore cannot factor every member of B_1, since $T \in B_1$ moves the integers. The smaller intervals are valid after rescaling the entire construction by $q \mapsto q/2$: use the half-unit translation $R(q) = q + 1/2$, the bound $B_{1/2}$, and the coding blocks $(2n - 1/2, 2n + 3/2)$. Equivalently, normalize the orbital action to R rather than T. The numerical scale is not intrinsic; its relation to the translation step is what matters.

4 The orbital sandwich theorem

Let $u \in \mathrm{Aut}(\mathbb{Q}, <)$. An *orbital* of u is the convex hull in $\mathbb{Q}$ of one cyclic orbit $\{qu^n : n \in \mathbb{Z}\}$. A nontrivial orbital is a convex subset on which u moves every point in one direction.

Lemma 4.1 (Orbital normalization). *Let O be a nontrivial orbital of u. There is a bijection $\theta_O : O \to \mathbb{Q}$, increasing when u moves to the right and decreasing when u moves to the left, such that*

$$(u|_O)^{\theta_O} = T. \tag{8}$$

Proof. Suppose first that $xu > x$ on O. Choose $x_0 \in O$ and an order isomorphism

$$[x_0, x_0u) \cap \mathbb{Q} \longrightarrow [0, 1) \cap \mathbb{Q}$$

taking x_0 to 0. Extend it equivariantly, mapping $[x_0u^n, x_0u^{n+1})$ onto $[n, n+1)$ for every $n \in \mathbb{Z}$. The resulting increasing bijection conjugates $u|_O$ to T.

If u moves to the left, apply the preceding construction to u^{-1} and then compose with the order-reversing map $q \mapsto -q$. This gives a decreasing bijection for which (8) holds. □

Theorem 4.2 (Orbital sandwich theorem)**.** *Let $u, v \in W$ satisfy*

$$u \wedge 1 \leq v \leq u \vee 1. \tag{9}$$

If $H \leq W$ is isolated and $u \in H$, then $v \in H$.

Proof. If $u = 1$, then (9) gives $v = 1$, so assume $u \neq 1$. The nontrivial orbitals of u, together with its singleton fixed points, form a partition of $\mathbb{Q}$ into ordered convex pieces. Condition (9) forces v to fix every point of $\mathrm{Fix}(u)$ and to preserve each nontrivial orbital. Let O be such an orbital and choose θ_O as in Lemma 4.1. Under this identification, write $\bar{v}_O = (v|_O)^{\theta_O}$. Whether θ_O is increasing or decreasing, (9) becomes

$$q \leq q\bar{v}_O \leq q + 1 \qquad (q \in \mathbb{Q}),$$

so $\bar{v}_O \in B_1$.

Use the fixed exponent list from Lemma 3.1 to choose $s_{j,O} \in S$ such that

$$\bar{v}_O = s_{1,O}^{T^{r_1}} \cdots s_{8,O}^{T^{r_8}}. \tag{10}$$

For each j, transport $s_{j,O}$ back to O and patch these maps over all nontrivial orbitals, taking the identity on $\mathrm{Fix}(u)$. Denote the resulting map by σ_j. On each piece it is an increasing bijection of that piece onto itself (also when θ_O is decreasing). Since the pieces form an ordered convex partition, the patch is a global increasing bijection of $\mathbb{Q}$.

Put $M = \|u - \mathrm{id}\|_\infty$. By (3), for every $q \in \mathbb{Q}$ the point $q\sigma_j$ lies in the interval with endpoints qu^{-2} and qu^2; this statement is unchanged when θ_O is decreasing. Since

$$|qu^{\pm 2} - q| \leq 2M,$$

we obtain $\|\sigma_j - \mathrm{id}\|_\infty \leq 2M$, and hence $\sigma_j \in W$. Transporting (10) back on every orbital gives the global identity

$$v = \sigma_1^{u^{r_1}} \cdots \sigma_8^{u^{r_8}}. \tag{11}$$

It remains to prove that every σ_j lies in H.

Fix j. On each nontrivial orbital O, apply Lemma 3.2 to $s_{j,O}$. Transport the resulting four maps back to O and patch over the orbitals, again using the identity on $\mathrm{Fix}(u)$. Call the global maps A, B, C, D. A local displacement bound of 4 means that, for every q, each of qA, qB, qC, qD lies in the interval with endpoints qu^{-4} and qu^4. Consequently,

$$\|A - \mathrm{id}\|_\infty,\ \|B - \mathrm{id}\|_\infty,\ \|C - \mathrm{id}\|_\infty,\ \|D - \mathrm{id}\|_\infty \leq 4M,$$

so $A, B, C, D \in W$. The local identities patch to

$$B^{u^4} = B^{-1}, \qquad C^{u^4} = C^{-1}, \qquad D^{u^4} = D^{-1}, \tag{12}$$

$$(AB)^2 = B^2, \tag{13}$$

$$\sigma_j = [A^{CD}, A^{C^{-1}}]. \tag{14}$$

We now use Lemma 2.5 with $t = u^4$. Explicitly, the first identity in (12) also gives $u^4 B u^{-4} = B^{-1}$, and hence

$$(Bu^4)^2 = B(u^4 B u^{-4})u^8 = u^8 \in H.$$

Since H is isolated, $Bu^4 \in H$, and therefore $B \in H$. The same argument gives $C, D \in H$. Equation (13) now gives $(AB)^2 = B^2 \in H$, so isolation yields $AB \in H$, whence $A \in H$. Finally, (14) gives $\sigma_j \in H$.

Thus every factor in (11) belongs to H; since $u \in H$, that factorization gives $v \in H$. $\square$

5 Classification of the relatively convex subgroups

Definition 5.1. A subgroup C of an ℓ-group is a *convex ℓ-subgroup* if it is closed under $\wedge$ and $\vee$ and is order-convex: whenever $a, b \in C$ and $a \le x \le b$, one has $x \in C$.

Proposition 5.2. *Every isolated subgroup of W is a convex ℓ-subgroup.*

Proof. Let $H \le W$ be isolated and let $h \in H$. Both $h \vee 1$ and $h \wedge 1$ lie between $h \wedge 1$ and $h \vee 1$, so Theorem 4.2 gives

$$h \vee 1 \in H, \qquad h \wedge 1 \in H.$$

For $h, k \in H$, bi-invariance of the lattice order gives

$$h \vee k = ((hk^{-1}) \vee 1)k \in H, \qquad h \wedge k = ((hk^{-1}) \wedge 1)k \in H.$$

Thus H is an ℓ-subgroup.

Now suppose $a, b \in H$ and $a \le g \le b$. Left multiplication by a^{-1} gives

$$1 \le a^{-1}g \le a^{-1}b.$$

Apply Theorem 4.2 with $u = a^{-1}b \in H$ and $v = a^{-1}g$. It follows that $a^{-1}g \in H$, hence $g \in H$. Therefore H is order-convex. $\square$

The first inclusion in the standard chain is valid for arbitrary ℓ-groups, not special to W: convex ℓ-subgroups are intersections of prime convex ℓ-subgroups, whose right-coset spaces are totally ordered [2, Section 2 and Theorem 2.1]. We recall the argument in the present notation.

Proposition 5.3. *Every convex ℓ-subgroup of an ℓ-group G is right-relatively convex. Thus* $\mathrm{Conv}_\ell(G) \subseteq \mathcal{R}(G)$ *for every ℓ-group G.*

Proof. The underlying group of any ℓ-group is right-orderable, by its classical order-permutation representation; see [2, p. 423]. Thus Lemma 2.2 applies.

Let C be a convex ℓ-subgroup of G. The case $C = G$ is immediate, so assume $C \ne G$ and let $x \notin C$. A union of a chain of convex ℓ-subgroups containing C and avoiding x is again such a subgroup. Thus Zorn's lemma gives a subgroup P_x maximal among them. Any larger convex ℓ-subgroup also contains C, so P_x is maximal among all convex ℓ-subgroups avoiding x. It is therefore regular and hence prime [2, Section 2, p. 424]. For a convex ℓ-subgroup P, primeness is equivalent to totality of the canonical order on the right cosets $P\backslash G$ [2, Theorem 2.1]. In multiplicative notation this order may be written

$$Pa \le_P Pb \iff pa \le b \text{ for some } p \in P.$$

It is invariant under right multiplication. Therefore Lemma 2.2 shows that every P_x is right-relatively convex. Since

$$C = \bigcap_{x \notin C} P_x,$$

and arbitrary intersections of right-relatively convex subgroups are right-relatively convex, $C \in \mathcal{R}(G)$. $\square$

Corollary 5.4. *For W,*

$$\mathcal{R}(W) = \mathrm{Isol}(W) = \mathrm{Conv}_\ell(W).$$

Proof. Apply the standard chain

$$\mathrm{Conv}_\ell(W) \subseteq \mathcal{R}(W) \subseteq \mathrm{Isol}(W).$$

Proposition 5.2 supplies the additional inclusion $\mathrm{Isol}(W) \subseteq \mathrm{Conv}_\ell(W)$. Thus all three subgroup families are equal. $\square$

6 Distributive, but not a chain

The convex ℓ-subgroups of every ℓ-group form a distributive lattice under inclusion [3, Sections 2–3]. The equality in Corollary 5.4 is an equality of subgroup families with the same inclusion order, so their lattice operations also coincide. Hence $\mathcal{R}(W)$ is distributive.

It remains to show that it is not linearly ordered. Define

$$\begin{aligned} C_{\mathrm{R}} &= \{g \in W : g \text{ fixes pointwise some right ray}\}, \\ C_{\mathrm{L}} &= \{g \in W : g \text{ fixes pointwise some left ray}\}. \end{aligned}$$

Equivalently, $\mathrm{supp}(g)$ is bounded above for $g \in C_{\mathrm{R}}$ and bounded below for $g \in C_{\mathrm{L}}$.

Lemma 6.1. *The subgroups C_{R} and C_{L} are convex ℓ-subgroups of W.*

Proof. If two automorphisms fix a sufficiently far right ray, then so do their product, their inverses, their pointwise minimum, and their pointwise maximum. Hence C_{R} is an ℓ-subgroup. If $f, g \in C_{\mathrm{R}}$ and $f \le h \le g$, choose a right ray on which both f and g are the identity. On that ray, $q \le qh \le q$, so h is also the identity there. Thus C_{R} is order-convex. The proof for C_{L} is symmetric. □

Consider the order automorphism λ defined by

$$q\lambda = \begin{cases} q+1, & q \le -1, \\ \dfrac{q+1}{2}, & -1 \le q \le 1, \\ q, & q \ge 1. \end{cases} \tag{15}$$

The formulas agree at the endpoints. The map is a strictly increasing bijection of $\mathbb{Q}$, has displacement at most 1, and

$$\mathrm{supp}(\lambda) = (-\infty, 1) \cap \mathbb{Q}.$$

Therefore $\lambda \in C_{\mathrm{R}} \setminus C_{\mathrm{L}}$. Define

$$q\rho = -((-q)\lambda).$$

Then $\rho \in W$ and $\mathrm{supp}(\rho) = (-1, \infty) \cap \mathbb{Q}$, so $\rho \in C_{\mathrm{L}} \setminus C_{\mathrm{R}}$. Hence the two convex ℓ-subgroups in Lemma 6.1 are incomparable. By Corollary 5.4, they belong to $\mathcal{R}(W)$. Thus $\mathcal{R}(W)$ is not a chain, and Theorem 1.1 follows.

A A direct proof of the bounded coding lemma

The construction below is adapted from the auxiliary-map construction in [4, Section 4]. We keep the four maps separate, use only one input automorphism, and use adjacent blocks rather than every twelfth block. All identities and displacement bounds needed in Lemma 3.2 are proved here.

A.1 An elementary commutator fact

Lemma A.1. *Every order automorphism of a countable dense linear order without endpoints is a commutator.*

Proof. By an order isomorphism, it suffices to work on $\mathbb{Q}$. Given $g \in \mathrm{Aut}(\mathbb{Q}, <)$, put

$$t = T \vee (Tg^{-1}).$$

Then $t \geq T$ and $tg \geq T$. An automorphism $v \geq T$ has just one orbital, since $qv^n \geq q+n$ and $qv^{-n} \leq q-n$ for $n \geq 0$. By Lemma 4.1, both t and tg are conjugate to T. Choose k with $t^k = tg$. It follows that

$$[t,k] = t^{-1}t^k = g.$$

□

A.2 Blocks and coordinates

For $n \in \mathbb{Z}$ let

$$K_n = (4n-1, 4n+3) \cap \mathbb{Q}, \qquad J_n = (4n, 4n+2) \cap \mathbb{Q}.$$

The K_n, together with the singleton points in $4\mathbb{Z}-1$, partition $\mathbb{Q}$. Let

$$L = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Q}$$

with its lexicographic order, and put

$$E = \{(0,0,x) : -1 < x < 1,\ x \in \mathbb{Q}\}.$$

Choose an order isomorphism $\theta_0 : K_0 \to L$ taking 0 to $(0,0,-1)$ and 2 to $(0,0,1)$. Such an isomorphism is obtained by choosing order isomorphisms on the three open intervals separated by these two points and then mapping the two points as prescribed. In particular, $J_0\theta_0 = E$. Define

$$q\theta_n = (q-4n)\theta_0 \qquad (q \in K_n).$$

Then $J_n\theta_n = E$ and

$$(qT^4)\theta_{n+1} = q\theta_n \qquad (q \in K_n). \tag{16}$$

Thus translation by 4 advances the block index by one without changing its internal coordinates.

Fix $h \in S$. Under θ_n, the restriction $h|_{K_n}$ is supported in E. Identify E with $(-1,1)\cap\mathbb{Q}$ using its last coordinate, and write h_n for the induced automorphism. By Lemma A.1, choose $k_{-1,n}, k_{1,n} \in \mathrm{Aut}((-1,1)\cap\mathbb{Q}, <)$ such that

$$[k_{-1,n}, k_{1,n}] = h_n. \tag{17}$$

Extend these two maps to $\mathbb{Q}$ by the identity outside $(-1,1)$, and set $k_{r,n} = 1$ for every integer $r \notin \{-1,1\}$.

A.3 The four maps and their bounds

Put $\varepsilon_n = (-1)^n$. Define $a_n, b_n, c_n, d_n \in \mathrm{Aut}(L,<)$ by

$$(j,m,x)a_n = \big(j,m,xk_{\varepsilon_n m,n}^{(-1)^j}\big), \tag{18}$$

$$(j,m,x)b_n = (j+\varepsilon_n, m, x), \tag{19}$$

$$(j,m,x)c_n = (j, m+\varepsilon_n, x), \tag{20}$$

$$(j,m,x)d_n = \begin{cases} (j,m,x), & (j,m) = (0,0), \\ (j,m,x+2\varepsilon_n), & (j,m) \neq (0,0). \end{cases} \tag{21}$$

The maps a_n, d_n preserve every fiber $\{(j,m)\} \times \mathbb{Q}$ and act on it by an increasing bijection. The maps b_n, c_n translate one of the integer coordinates. Hence all four are increasing bijections of L.

Transport them to K_n using θ_n^{-1}, and patch over $n \in \mathbb{Z}$, fixing every point of $4\mathbb{Z} - 1$. Denote the resulting automorphisms of $\mathbb{Q}$ by a, b, c, d. Each preserves every K_n, an interval of diameter 4, so

$$\|a - \mathrm{id}\|_\infty,\ \|b - \mathrm{id}\|_\infty,\ \|c - \mathrm{id}\|_\infty,\ \|d - \mathrm{id}\|_\infty \leq 4. \tag{22}$$

No bound on the coordinate displacement of the maps $k_{r,n}$ is needed: all their transported actions remain inside the same bounded block.

A.4 Inversion and square identities

By (16), the restriction of b^{T^4} to K_n, expressed in θ_n-coordinates, is b_{n-1}. Since $\varepsilon_{n-1} = -\varepsilon_n$, we have $b_{n-1} = b_n^{-1}$. Exactly the same argument applies to c_n, d_n. All maps fix the block endpoints, so globally

$$b^{T^4} = b^{-1}, \qquad c^{T^4} = c^{-1}, \qquad d^{T^4} = d^{-1}.$$

Translation of j by ε_n reverses its parity. In two successive applications of $a_n b_n$, the last-coordinate maps are therefore

$$k_{\varepsilon_n m,n}^{(-1)^j} \quad \text{and} \quad k_{\varepsilon_n m,n}^{(-1)^{j+\varepsilon_n}},$$

which are inverse to one another. Thus $(a_n b_n)^2 = b_n^2$ on every block, and $(ab)^2 = b^2$ globally.

A.5 Recovery of the input automorphism

Fix n and put

$$P_n = a_n^{c_n d_n}, \qquad Q_n = a_n^{c_n^{-1}}.$$

Both maps preserve every fiber $\{(j, m)\} \times \mathbb{Q}$. On the central fiber $\{(0, 0)\} \times \mathbb{Q}$, the map d_n is the identity, and (18)–(21) give

$$(0, 0, x)P_n = (0, 0, xk_{-1,n}), \tag{23}$$

$$(0, 0, x)Q_n = (0, 0, xk_{1,n}). \tag{24}$$

Indeed, c_n^{-1} first changes $m = 0$ to $m = -\varepsilon_n$ in the calculation of P_n, giving the index -1 in (18). For $Q_n = c_n a_n c_n^{-1}$, the index is 1. These statements hold for every $x \in \mathbb{Q}$, since the $k_{r,n}$ were extended by the identity.

On a noncentral fiber, the support of Q_n is contained in $\{(j, m)\} \times (-1, 1)$, whereas the support of P_n is contained in

$$\{(j, m)\} \times (-1 + 2\varepsilon_n,\ 1 + 2\varepsilon_n).$$

To see this, conjugation by c_n changes only the integer coordinate m, and the subsequent conjugation by d_n shifts the support in the last coordinate by $2\varepsilon_n$ on a noncentral fiber. The two open intervals are disjoint, for either sign of ε_n. The restrictions of P_n, Q_n to that fiber consequently commute.

It follows that $[P_n, Q_n]$ is the identity off E, and on E it equals $[k_{-1,n}, k_{1,n}] = h_n$ by (17). Transporting back to every K_n and noting that all maps fix the block endpoints gives

$$[a^{cd}, a^{c^{-1}}] = h.$$

Together with (22) and the preceding identities, this proves Lemma 3.2.

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