A COUNTEREXAMPLE TO PROBLEM 21.99 OF THE KOUROVKA NOTEBOOK

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Abstract. We construct a finite solvable group with a faithful transitive action of degree 23 437 500 and distinct points α, β such that every element taking α to β fixes exactly one point. This gives a negative answer to Problem 21.99 of the Kourovka Notebook. The group is an extension of $\mathbb{F}_5^{12}$ by a group of order 192. The fixed-point calculation is proved directly on twelve affine blocks.

1. Introduction

Question. The following problem appears in the Kourovka Notebook [1, p. 182].

> **21.99.** *Conjecture:* If G is a transitive permutation group on a finite set Ω, then for any distinct α, β in Ω there is an element $g \in G$ with $\alpha^g = \beta$ whose number of fixed points is different from 1.
>
> *P. Müller*

The same question is stated in [2, Remark (f)]. Here $\mathrm{Fix}_\Omega(g)$ denotes the set of points fixed by g. We give a counterexample, already within the class of solvable groups.

Theorem 1. *There is a solvable group G of order $46\,875\,000\,000$ with a faithful transitive action on a set Ω of size $23\,437\,500$ and distinct points $\alpha, \beta \in \Omega$ such that*

$$|\mathrm{Fix}_\Omega(h)| = 1 \qquad \text{for every } h \in G \text{ with } \alpha^h = \beta.$$

The set of such transporters has $2\,000$ elements.

The construction uses two six-dimensional modules for a group of order 192. For each prescribed transporter, four of the twelve normal blocks are invariant. One module rules out two of these blocks. On the other two, the second module gives exactly one fixed point on one block, while the vector $(1, 2) \in \mathbb{F}_5^2$ obstructs a fixed point on the other. All actions are on the right.

2. The group and its modules

Let

$$P = \langle x, y, c \mid x^4 = y^4 = c^2 = 1,\ [x, y] = c,\ [x, c] = [y, c] = 1\rangle, \qquad [x, y] = x^{-1}y^{-1}xy.$$

An explicit model for P is $(\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$, with multiplication

$$(a, b, d)(a', b', d') = (a + a', b + b', d + d' + ba').$$

The first two coordinates are taken modulo 4, and the third modulo 2; the elements x, y, c correspond to $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. In particular, $|P| = 32$, and its elements have unique expressions $x^a y^b c^d$, where $0 \le a, b < 4$ and $0 \le d < 2$. Put $u = x^2$ and $v = y^2$. Then

$$C = Z(P) = \langle u, v, c\rangle \cong C_2^3, \qquad P' = \langle c\rangle.$$

Define automorphisms σ, τ by

$$\begin{array}{c|ccc} & x & y & c \\ \hline \sigma & y & (xy)^{-1} & c \\ \tau & xvc & yuv & c \end{array} \tag{1}$$

The coordinate model verifies that $\sigma^3 = \tau^2 = 1$ and $\sigma\tau = \tau\sigma$. Define $Q = P \rtimes \langle \sigma, \tau\rangle \cong P \rtimes C_6$, a group of order 192.

We use the convention $\sigma x \sigma^{-1} = y$, and likewise for the other entries in (1). The subgroup $S = \langle x, v, c\rangle$ is abelian of order 16. Representatives for $S\backslash Q$ are

$$r_{t,k,\varepsilon} = y^t \sigma^k \tau^\varepsilon, \qquad t, \varepsilon \in \{0,1\}, \quad k \in \{0,1,2\}. \tag{2}$$

Let U and W be vector spaces over $\mathbb{F}_5$ with respective bases $e_0, \dots, e_5$ and $f_0, \dots, f_5$. Set $V = U \oplus W$, and write $\iota = 2 \in \mathbb{F}_5$, so that $\iota^2 = -1$. Define right actions by

(3)

j	e_j^x	e_j^y	e_j^σ	e_j^τ	f_j^x	f_j^y
0	$-e_0$	e_3	e_1	e_3	f_0	ιf_0
1	e_4	e_4	e_2	e_4	$-f_1$	$-\iota f_1$
2	e_5	$-e_2$	e_0	e_5	ιf_2	$-\iota f_2$
3	e_3	e_0	e_4	e_0	$-\iota f_3$	$-\iota f_3$
4	e_1	$-e_1$	e_5	e_1	$-\iota f_4$	f_4
5	$-e_2$	e_5	e_3	e_2	$-\iota f_5$	$-f_5$

On the basis of W, the remaining generators act by the permutations

$$\sigma = (f_0\ f_2\ f_4)(f_1\ f_3\ f_5), \qquad \tau = (f_0\ f_1)(f_2\ f_3)(f_4\ f_5). \tag{4}$$

These operators satisfy the defining relations of P and (1); hence they define Q-modules. In particular, c acts as $-I$ on U and as I on W.

The three P-invariant summands of U are

$$U_1 = \mathrm{span}_{\mathbb{F}_5}\{e_0, e_3\}, \qquad U_2 = \mathrm{span}_{\mathbb{F}_5}\{e_1, e_4\}, \qquad U_3 = \mathrm{span}_{\mathbb{F}_5}\{e_2, e_5\}. \tag{5}$$

The signs of (u, v, c) on these summands are respectively $(+,+,-), (+,-,-), (-,+,-)$. For an interpretation of W, define the linear character

$$\chi : P \longrightarrow \mathbb{F}_5^*, \qquad \chi(x) = 1, \quad \chi(y) = \iota, \quad \chi(c) = 1.$$

Then W is the module induced from χ, using representatives $\sigma^k \tau^\varepsilon$ indexed by $2k+\varepsilon$. In particular, $f_0^s = \chi(s) f_0$ for $s \in P$.

The action of Q on V is faithful. Indeed, the permutation of the W-coordinates first forces any element of its kernel to belong to P. If $x^a y^b c^d$ acts trivially on U, commuting it with x and y shows that a, b are even, since c acts nontrivially. It is therefore central, and the three independent central characters in (5) force it to be the identity.

3. The coset action

Define $G = V \rtimes Q$ with multiplication

$$(q, z)(r, z') = (qr, z^r + z') \qquad (q, r \in Q,\ z, z' \in V). \tag{6}$$

Thus (q, z) acts on V as the affine map $b \mapsto b^q + z$. Put

$$A_U = \mathrm{span}_{\mathbb{F}_5}\{e_3, e_1 + e_4\}, \qquad A_W = \mathrm{span}_{\mathbb{F}_5}\{f_0\}, \qquad A = A_U \oplus A_W.$$

The table shows that A is S-invariant, so

$$H = \{(s, a) : s \in S,\ a \in A\} \le G$$

has order $16 \cdot 5^3 = 2\,000$. Let G act by right multiplication on $\Omega = H\backslash G$. Then

$$|G| = 192 \cdot 5^{12} = 46\,875\,000\,000, \qquad |\Omega| = 12 \cdot 5^9 = 23\,437\,500.$$

The group is solvable, because V and Q/P are abelian and P is nilpotent.

The action on Ω is faithful. In fact,

$$A^\sigma = \mathrm{span}_{\mathbb{F}_5}\{e_4, e_2 + e_5, f_2\}, \qquad A \cap A^\sigma = 0.$$

For $K = \mathrm{core}_G(H)$ this implies $K \cap V = 1$. Since K and V are normal in G, we have $[K, V] \le K \cap V = 1$. Faithfulness of the Q-module gives $C_G(V) = V$, whence $K \le V$ and finally $K = 1$.

Choose

$$w = e_2 + 2e_5 + \sum_{j=0}^{5} f_j, \qquad g = (y, w), \qquad \alpha = H, \quad \beta = Hg. \tag{7}$$

Since $y \notin S$, the points α, β are distinct. By (6), their complete transporter set is

$$Hg = \{(q, w + \ell^y) : q \in Sy,\ \ell \in A\}. \tag{8}$$

The V-orbits in Ω are indexed by $S\backslash Q$. The block represented by $r \in Q$ is identified with V/A^r by

$$z + A^r \longmapsto H(r, z).$$

It is preserved by (q, b) precisely when $rq = sr$ for some $s \in S$. In that case

$$H(r, z)(q, b) = H(rq, z^q + b) = H(r, z^q + b).$$

For a q-invariant subspace $X \le V$, write $\Delta_q X = \{z^q - z : z \in X\}$. The fixed-point equation on this block is therefore solvable if and only if

$$b \in A^r + \Delta_q V. \tag{9}$$

When it is solvable, the number of fixed points equals $|\ker(q - I \mid V/A^r)|$. The equation and its number of solutions separate across U and W.

4. The fixed-point calculation

Fix an element of (8). Write its quotient part as

$$q = x^a y^b c^d, \qquad 0 \le a < 4, \quad b \in \{1, 3\}, \quad d \in \{0, 1\}. \tag{10}$$

Conjugation by σ cycles the three nonzero elements xC, yC, xyC of P/C, while τ and inner conjugation act trivially on P/C. It follows that among (2), exactly four blocks are fixed by q: all choices of t, ε, with $k = 2$ if a is even and $k = 1$ if a is odd. All other blocks contribute no fixed points.

The W-component. The diagonal table (3) shows that q has exactly one fixed coordinate on W, as follows:

(11)

condition	fixed coordinate
$a = 0$	f_4
$a = 2$	f_5
$(a, b) = (1, 1)$ or $(3, 3)$	f_2
$(a, b) = (1, 3)$ or $(3, 1)$	f_3

Denote this coordinate by f_j. Since $j \notin \{0, 1\}$ and $A_W^y = A_W$, the f_j-coordinate of $w + \ell^y$ is always 1. For $r = r_{t,k,\varepsilon}$ we have

$$A_W^r = \mathrm{span}_{\mathbb{F}_5}\{f_{2k+\varepsilon}\}.$$

Exactly two of the four invariant blocks therefore satisfy $A_W^r = \mathrm{span}_{\mathbb{F}_5}\{f_j\}$, corresponding to the two values of t and the appropriate value of ε. On these two blocks, $q - I$ is invertible on W/A_W^r, so the W-equation has a unique solution for every offset.

On each of the other two invariant blocks, the f_j-coordinate survives in the quotient, q acts identically on it, and the offset is 1. The fixed-point equation is inconsistent. Thus only two blocks remain to be considered.

The U-component. On either remaining block put $s = rqr^{-1} \in S$. Since $A^r_W = \mathrm{span}_{\mathbb{F}_5}\{f_j\}$ and q fixes f_j, we have $\chi(s) = 1$. The element s is noncentral, so it has the form $x^h y^{2l} c^m$ with h odd. As $\chi(s) = (-1)^l$, the exponent l is even. Consequently

$$s = x^h c^m \qquad (h \text{ odd}). \tag{12}$$

Both generators of A_U are fixed by x and negated by c. It follows that q acts on A^r_U by the single scalar $(-1)^m$. The two remaining representatives are r_0 and yr_0, where $r_0 = \sigma^k \tau^\varepsilon$. Their normalized elements are s and $ysy^{-1} = sc$, by (12). Hence q acts as $+1$ on one of the two spaces A^r_U and as -1 on the other.

For every q in (10), we have

$$q^2 = u^a v c^{ab}.$$

Its signs on U_1, U_2, U_3 are $((-1)^a, -(-1)^a, 1)$. Thus $q^2 = -I$ on one summand and $q^2 = I$ on the other two. Moreover, q anticommutes with x on U, because b is odd and c acts as $-I$. On each summand where $q^2 = I$, the invertible operator x therefore interchanges the $+1$ and -1 eigenspaces of q, which both have dimension one. The remaining summand has no fixed vector. We conclude that

$$\dim C_U(q) = 2. \tag{13}$$

On the block where q acts as $+1$ on A^r_U, (13) gives $A^r_U = C_U(q)$. Since q has order 4, which is prime to 5, we have

$$U = C_U(q) \oplus \Delta_q U.$$

Thus $q - I$ is invertible on U/A^r_U. Together with the W-calculation, this block contributes exactly one fixed point for every offset in (8).

On the last block, q acts as -1 on A^r_U, so

$$A^r_U \le \Delta_q U. \tag{14}$$

The permitted corrections satisfy

$$A^y_U = \mathrm{span}_{\mathbb{F}_5}\{e_0, e_4 - e_1\} \le U_1 \oplus U_2.$$

Their projection to U_3 is zero. Therefore the U_3-component of the offset is always $(1, 2)$ in the ordered basis (e_2, e_5). On U_3, the operators q in (10) are precisely among

$$\pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The possible images $\Delta_q U_3$ are the four lines

$$\xi = 0, \qquad \eta = 0, \qquad \xi + \eta = 0, \qquad \xi - \eta = 0.$$

The vector $(1, 2)$ lies on none of these lines over $\mathbb{F}_5$. By (14), the U-component of the necessary condition (9) is therefore impossible on this block.

Exactly one block contributes a fixed point, and its contribution is one. This proves the assertion for every element of the full transporter set (8), and completes the proof of Theorem 1. □

Remark. The finite data were also checked by exact linear algebra in independent Python implementations. For each of the sixteen elements $q \in Sy$, one invariant block satisfies $A^r + \Delta_q V = V$; on each other invariant block, $w \notin A^r + \Delta_q V + A^y$. These checks confirm all 2 000 transporters simultaneously. The proof above is independent of the computations and requires no enumeration of Ω.

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References

[1] E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved Problems in Group Theory. The Kourovka Notebook*, 21st ed., Novosibirsk, 2026, version 46, Problem 21.99, p. 182. arXiv:1401.0300v46.

[2] P. Müller, *On transitive sets of derangements in primitive groups*, Arch. Math. **126** (2026), 551–556. doi:10.1007/s00013-026-02240-3.