A MONOMIAL COUNTEREXAMPLE TO THE LITERAL STATEMENT OF KOUROVKA NOTEBOOK 6.38(B)

Abstract. The invertible monomial subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_2)$ meets every conjugacy class and is not parabolic. We give its explicit matrix description and a proof covering the whole infinite group. This answers the general-linear assertion of Problem 6.38(b) as worded, using the dimension-two, characteristic-two family already mentioned in the Notebook archive.

1. Problem and construction

The general-linear assertion in Kourovka Notebook 6.38(b) asks:

> Is it true that an arbitrary subgroup of $\mathrm{GL}_n(k)$ that intersects every conjugacy class is parabolic?

It also asks how far the statement holds for other groups of Lie type. We answer the displayed assertion with $n = 2$ and $k = \overline{\mathbf{F}}_2$. The supplied wording excludes neither dimension two nor characteristic two. This does not classify other Lie types or address a reformulation excluding known exceptions. The characteristic-two monomial family is already mentioned in the archive entry for 6.38(a); no novelty is claimed. In the repository Notebook source `1401.0300v45.pdf`, the question and archive entry are on printed pages 19 and 221, respectively.

Write $k^* = k \setminus \{0\}$ and $G = \mathrm{GL}_2(k)$. For $a, b \in k^*$ put

$$D(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad A(a,b) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

and define

$$H = \{D(a,b) : a, b \in k^*\} \;\cup\; \{A(a,b) : a, b \in k^*\}.$$

Theorem 1. *The set H is a proper subgroup of G, meets every conjugacy class of G, and is not parabolic.*

2. Proof

The subgroup conditions. All arithmetic is in characteristic two. Both matrix types have determinant $ab \neq 0$, and the identity is $D(1,1)$. For $a, b, c, d \in k^*$, multiplication gives

$$\begin{aligned} D(a,b)D(c,d) &= D(ac,bd), & D(a,b)A(c,d) &= A(ac,bd), \\ A(a,b)D(c,d) &= A(ad,bc), & A(a,b)A(c,d) &= D(ad,bc). \end{aligned}$$

Their inverses are $D(a^{-1}, b^{-1})$ and $A(b^{-1}, a^{-1})$, respectively. Therefore $H \leq G$.

Every conjugacy class meets H. Let $X \in G$. Its characteristic polynomial splits over k, and its eigenvalues are nonzero. If X is diagonalizable, it is conjugate to some $D(a,b) \in H$. This case includes scalar matrices.

Otherwise its characteristic polynomial is $(T-\lambda)^2$ for some $\lambda \in k^*$: distinct eigenvalues would provide two independent eigenvectors. Set $N = X - \lambda I$. The 2×2 Cayley–Hamilton identity gives $N^2 = 0$, and $N \neq 0$. Choose v with $w = Nv \neq 0$. The vectors w, v are independent: if $w = cv$, then $0 = Nw = cw$, so $c = 0$, contradicting $w \neq 0$. In the ordered basis (w, v), the matrix of X is

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Now define

$$M_\lambda = \begin{pmatrix} 0 & \lambda^2 \\ 1 & 0 \end{pmatrix} \in H, \qquad Q_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}.$$

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We have $\det Q_\lambda = 1$ and

$$Q_\lambda^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}.$$

Direct multiplication gives

$$M_\lambda Q_\lambda = \begin{pmatrix} \lambda^2 & 0 \\ \lambda & 1 \end{pmatrix}, \qquad Q_\lambda^{-1} M_\lambda Q_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = J_\lambda.$$

The lower-left entry in the last product is $\lambda^2 + \lambda^2 = 0$. Thus X is conjugate in G to M_λ. The diagonalizable and nondiagonalizable cases exhaust G.

Failure of parabolicity. The matrix

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is invertible and is neither diagonal nor antidiagonal, so $H \neq G$.

Choose $\omega \in k$ with $\omega^2 + \omega + 1 = 0$. It exists because k is algebraically closed, and $\omega \notin \{0, 1\}$. Both

$$D(\omega, 1), \qquad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

belong to H. A line invariant under $D(\omega, 1)$ is spanned by an eigenvector. The eigenvalues $\omega, 1$ are distinct, so the only such lines are $k(1, 0)$ and $k(0, 1)$. Explicitly, invariance of a line spanned by $(x, y) \neq (0, 0)$ forces $(1 + \omega)xy = 0$, hence $x = 0$ or $y = 0$. But W interchanges these coordinate lines. Thus H has no invariant line in k^2.

Parabolic subgroups of $\mathrm{GL}(V)$ over an algebraically closed field are stabilizers of flags of vector subspaces, with the trivial flag giving the whole group. In dimension two a nontrivial proper flag has the form $0 < L < V$ with $\dim L = 1$. Consequently every proper parabolic stabilizes a line. The proper subgroup H stabilizes no line and therefore is not parabolic. This proves the theorem.

Methods and provenance

This draft reorganizes the preserved certificate `source/counterexample.txt`. The proof uses elementary matrix identities, algebraic closedness, the 2×2 Cayley–Hamilton identity, and the flag description of general-linear parabolics. Its assertions about the infinite field follow from the written argument; no computed count or finite enumeration is a proof input.

The archived solver was configured as `gpt-6-astra`; the campaign validator recorded PASS. That historical record is neither a fresh independent verdict nor human approval of this export. The scoped review of this rewritten proof is recorded separately; no two-blind-verifier qualification or repository status change is asserted.

The unchanged standard-library program `verify_counterexample.py` checks the conjugation and inverse identities over $\mathbf{F}_2[t]$ and the displayed properness and invariant-line witnesses over $\mathbf{F}_4$. The replay and source-integrity records accompany the package. These are supporting identity checks and do not enumerate G. GAP was unavailable on the export host, and no GAP reproduction is claimed. The proof does not rely on the checker.