INTERSECTIONS OF GOOD SUBNORMAL SUBGROUPS: A COUNTEREXAMPLE TO KOUROVKA NOTEBOOK PROBLEM 8.74

A. IVANOV

Abstract. A subnormal subgroup is called good if its join with every subnormal subgroup is subnormal. We construct two good subnormal subgroups whose intersection is not good, giving a negative answer to Problem 8.74 of the Kourovka Notebook. The ambient group is a countable locally finite 2-group of derived length at most three. Goodness is proved by an explicit augmentation-ideal bound; the argument does not rely on computation.

1. The problem and the result

All groups are abstract groups, and subnormality means subnormality along a finite chain. Following Problem 8.74, proposed by J. G. Thompson in the Kourovka Notebook [1, Problem 8.74, p. 30], a subgroup $H \lhd\lhd G$ is *good* if

$$\langle H, R\rangle \lhd\lhd G \qquad \text{for every } R \lhd\lhd G.$$

The problem asks whether the intersection of two good subnormal subgroups must be good.

Theorem 1. *There is a countable locally finite 2-group G, of derived length at most three, with good subnormal subgroups H, K and a subnormal subgroup J such that*

$$\langle H \cap K, J\rangle \not\lhd\lhd G.$$

In particular, the answer to Problem 8.74 is negative.

The construction starts with a proper complement $P = \langle U, V\rangle$ that normally generates G. Two automorphic enlargements of U are good, but intersect in U itself. The main point is to prove goodness for an arbitrary subnormal subgroup, not just for subgroups of a prescribed form.

2. The ring, module, and group

Put

$$E = \mathbb{F}_2[t_i \mid i \in \mathbb{Z}]/(t_i^2 \mid i \in \mathbb{Z}), \qquad E_0 = \mathbb{F}_2[t_i \mid i \neq 0]/(t_i^2 \mid i \neq 0).$$

Every polynomial involves only finitely many variables. Every element c of the augmentation ideal of E satisfies $c^2 = 0$, so $(1+c)^2 = 1$.

Let $\mathcal{X}$ be the set of subsets $X \subseteq \mathbb{Z}$ containing every sufficiently negative integer and no sufficiently positive integer. Let D be the $\mathbb{F}_2$-vector space with basis $\{e_X : X \in \mathcal{X}\}$. Define a right E-module structure by

$$e_X t_i = \begin{cases} e_{X\cup\{i\}}, & i \notin X, \\ 0, & i \in X. \end{cases}$$

These operators commute and have square zero. Let D_0 and D_1 be the spans of the basis vectors with $0 \notin X$ and $0 \in X$, respectively. Then

$$D = D_0 \oplus D_1, \qquad D_0 t_0 = D_1, \qquad D_1 t_0 = 0, \tag{1}$$

and multiplication by t_0 is a vector-space isomorphism $D_0 \longrightarrow D_1$. Both summands are E_0-modules.

Date: 11 September 2026.

Key words and phrases. Subnormal subgroup, good subnormal subgroup, locally finite group, augmentation ideal.

Set $M = D \oplus D$, written additively as row vectors. For linear combinations a, b of the variables t_i, set

$$u(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad v(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

Write $u_i = u(t_i)$ and $v_i = v(t_i)$, and put

$$U = \langle u_i : i \in \mathbb{Z} \rangle, \qquad V = \langle v_i : i \in \mathbb{Z} \rangle, \qquad P = \langle U, V \rangle \leq \mathrm{GL}_2(E).$$

The matrices act on M on the right. Define $G = M \rtimes P$, with $m^p = mp$ for $m \in M$ and $p \in P$. Thus, representing an element by $pm = (p, m)$, multiplication is

$$(p, m)(q, n) = (pq, mq + n).$$

We use the commutator convention $[x, y] = x^{-1}y^{-1}xy$. Since $a^2 = b^2 = 0$, matrix multiplication gives

$$[u(a), v(b)] = (1 + ab)I_2. \tag{2}$$

The groups U and V are abelian of exponent two. Consequently, P has class at most two and P' consists of scalar matrices. By collecting the U- and V-factors, every $w \in P$ has an expression

$$w = s_w u(a_w) v(b_w), \tag{3}$$

where a_w, b_w are linear combinations of the t_i and s_w is a scalar unit with constant coefficient one. In particular, $s_w^2 = 1$.

Define

$$A = D \oplus 0, \quad B = 0 \oplus D, \quad B_0 = 0 \oplus D_0, \quad B_1 = 0 \oplus D_1,$$

and set

$$H = UB_0, \qquad K = U\big(B_0(1 + t_0)\big), \qquad J = V. \tag{4}$$

3. Subnormality and the intersection

Put $F = M \rtimes (UP')$. Since $UP' \triangleleft P$, we have $F \triangleleft G$. The group UB is abelian: U is abelian and fixes B pointwise. It is also normal in F. Indeed, conjugating U by M adds elements of B, while P' centralizes U and preserves B. Hence

$$H \triangleleft UB \triangleleft F \triangleleft G, \qquad U \triangleleft UB \triangleleft F \triangleleft G. \tag{5}$$

Interchanging the two coordinates gives

$$V \triangleleft VA \triangleleft M \rtimes (VP') \triangleleft G. \tag{6}$$

Multiplication by $1 + t_0$ on M is an involutory automorphism commuting with every element of P. It extends to an automorphism ϕ of G fixing P pointwise, and $K = \phi(H)$. In particular, K is subnormal. Moreover,

$$B_0 \cap B_0(1 + t_0) = 0.$$

To see this, if $f \in D_0$ and $f + ft_0 \in D_0$, then $ft_0 = 0$ by (1); injectivity of multiplication by t_0 on D_0 gives $f = 0$. Uniqueness of the P- and M-coordinates now yields

$$H \cap K = U. \tag{7}$$

For each $X \in \mathcal{X}$, choose $i \in X$ and put $Y = X \setminus \{i\}$. Then $Y \in \mathcal{X}$ and $e_Y t_i = e_X$. The action formulas therefore give

$$[M, U] = B, \qquad [M, V] = A. \tag{8}$$

The normal closure of P in G contains M, by (8), and hence equals G. But P is proper, since $M \neq 0$. A proper subnormal subgroup cannot have normal closure equal to the whole group: a subnormal chain with repetitions removed has a proper penultimate normal term containing that subgroup. Thus

$$\langle H \cap K, J \rangle = \langle U, V \rangle = P \not\!\triangleleft\!\triangleleft\, G. \tag{9}$$

It remains to show that H and K are good.

4. The goodness argument

If a group Q acts on an $\mathbb{F}_2$-vector space L, write $\Delta(Q)$ for the augmentation ideal of $\mathbb{F}_2[Q]$. Thus $L\Delta(Q)^r$ is spanned by

$$x(q_1-1)\cdots(q_r-1), \qquad x\in L, \quad q_1,\ldots,q_r\in Q.$$

Proposition 2. *If $R \lhd\lhd G$ has a subnormal chain of length $d\geq 1$, then $\langle H,R\rangle$ has a subnormal chain in G of length at most $3d+4$. In particular, H is good.*

Proof. Let $\pi : G \longrightarrow P$ be the projection and put

$$W=\pi(R), \qquad L=\langle H,R\rangle, \qquad Q=\langle U,W\rangle=\pi(L), \qquad T=L\cap M.$$

Then T is Q-invariant and contains B_0. From a subnormal chain for R we obtain

$$M\Delta(W)^d=[M,{}_dR]\leq R\cap M\leq T. \tag{10}$$

Indeed, each successive commutator with R descends one term of the chain. The action on M depends only on the projection W, because M is abelian. No splitting assumption on R is used.

Put

$$N=Mt_0=D_1\oplus D_1, \qquad A_1=D_1\oplus 0, \qquad T_0=B_0\mathbb{F}_2[Q]\leq T.$$

The space T_0 is an E_0-module, since B_0 is an E_0-module and scalar multiplication by E_0 commutes with Q. We will only use this scalar-closure property for T_0, not for T.

For coefficients in (3), a bar denotes evaluation at $t_0=0$. Let $\mathfrak{b}$ be the ideal of E_0 generated by $\bar b_w$ for $w\in W$. We first prove

$$B_1\mathfrak{b}\leq T_0, \qquad A_1\mathfrak{b}^2\leq T_0. \tag{11}$$

For $w\in W$, formula (2) gives

$$[u_0,w]=(1+t_0b_w)I_2=(1+t_0\bar b_w)I_2.$$

Since $u_0,w\in Q$, applying $[u_0,w]-1$ to B_0 gives $B_1\bar b_w\leq T_0$. The E_0-module property of T_0 gives the first inclusion in (11).

For $w,w'\in W$ and $f\in D_1$, formula (3) and $D_1t_0=0$ give

$$(0,f\bar b_w)w'=\big(f\bar b_w\bar s_{w'}\bar b_{w'},\, f\bar b_w\bar s_{w'}\big).$$

The vector consisting only of the second coordinate belongs to T_0 by the first inclusion in (11). Subtracting it and multiplying by the scalar unit $\bar s_{w'}^{-1}\in E_0$ gives

$$(f\bar b_w\bar b_{w'},0)\in T_0.$$

Again using scalar closure gives $A_1\mathfrak{b}^2\leq T_0$.

We next show that

$$N\Delta(Q')^2\leq T_0. \tag{12}$$

Every element of Q' is scalar, because $Q\leq P$. Reduce the matrices in Q first modulo t_0, then modulo $\mathfrak{b}$. The generators from U become upper unitriangular matrices, and those from W become scalar multiples of upper unitriangular matrices. These all commute. Hence, for every $z\in Q'$, its reduction modulo t_0 has the form

$$\bar z=(1+c_z)I_2, \qquad c_z\in\mathfrak{b}.$$

On N, multiplication by t_0 is zero. A product of two augmentation operators from Q' therefore sends N into

$$A_1\mathfrak{b}^2+B_1\mathfrak{b}^2\leq T_0,$$

where the final inclusion follows from (11). This proves (12).

Consider the Q-module $\overline{N}=N/(N\cap T)$ and its filtration

$$\overline{N}\supseteq\overline{N}\Delta(Q')\supseteq 0. \tag{13}$$

Both terms are Q-invariant, and (12) implies that Q' acts trivially on both factors. Thus the operators induced by Q on either factor commute.

On all of M we have

$$(u(a)-1)(u(a')-1)=0,$$

so the U-augmentation operators have square zero. By (10), the W-augmentation operators have dth power zero on M/T, and hence on each factor of (13). In the commutative operator algebra on either factor, let I_U and I_W be the ideals generated by these two sets of operators. Then

$$I_U^2=0, \qquad I_W^d=0.$$

Since $Q=\langle U,W\rangle$, its augmentation operators lie in I_U+I_W. Expanding $(I_U+I_W)^{d+1}$, every term has either two I_U-factors or d I_W-factors. Consequently, $\Delta(Q)^{d+1}$ annihilates each of the two factors in (13), and therefore

$$N\Delta(Q)^{2(d+1)} \le T. \tag{14}$$

The action of U on $M/(T+N)$ is trivial, since

$$M\Delta(U) \le B = B_0+B_1 \le T+N.$$

The action of $Q=\langle U,W\rangle$ on this quotient is thus its W-action. Applying (10) gives

$$M\Delta(Q)^d \le T+N.$$

Combining this with (14) and Q-invariance of T, we obtain

$$M\Delta(Q)^{3d+2} \le T. \tag{15}$$

Finally, put $M_k = M\Delta(Q)^k$ and $\ell = 3d+2$. Each M_k is Q-invariant. Since $M_\ell \le T \le L$, equation (15) gives

$$L = LM_\ell \lhd LM_{\ell-1} \lhd \cdots \lhd LM_0 = ML = M \rtimes Q.$$

Normality at each step follows from $[M_k, L] \le M_{k+1}$ and the abelianness of M. As P has class at most two,

$$Q \lhd QP' \lhd P.$$

Taking inverse images under π adds at most two steps, proving the claimed bound $3d+4$. □

Completion of the proof of Theorem 1. Proposition 2 proves that H is good; the case $R=G$ is also immediate. Automorphisms preserve subnormality and joins, so $K=\phi(H)$ is good. The subgroup $J=V$ is subnormal by (6), whereas (9) gives the required failure of subnormality.

It remains to record the asserted properties of G. The set $\mathcal{X}$ is countable, because its members differ from the negative half-line in only finitely many places. Hence D, E, and G are countable. A finite set in P is contained in a subgroup generated by finitely many u_i, v_i. Such a group has class at most two, is generated by involutions, and has central commutators of order at most two; collecting generators and commutators shows that it is a finite 2-group. Thus P is locally finite. Given finitely many elements of G, their P-coordinates lie in a finite 2-subgroup P_0, and the $\mathbb{F}_2$-span of the P_0-orbits of their M-coordinates is finite and P_0-invariant. The given elements therefore lie in a finite 2-subgroup of G. Finally, M is abelian and P is metabelian, so G has derived length at most three. □

Acknowledgment. This work was carried out with assistance from ChatGPT.

References

[1] E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved Problems in Group Theory. The Kourovka Notebook*, 21st ed., Novosibirsk, 2026; arXiv:1401.0300v46, 1 September 2026.