FINITE GENERATORS FOR CLASSES OF GROUP ISOTOPES: NEGATIVE ANSWERS TO KOUROVKA PROBLEM 9.4

Abstract. The isotopic closure of the variety or quasivariety generated by any nontrivial finite group cannot be generated by a finite quasigroup. For pseudovarieties defined by disjunctive identities, as in the original source, the cyclic group of order three is a counterexample. The proofs use a finite permutation identity and a finite disjunction separating nonisomorphic quasigroups.

1. The problem and the result

Problem 9.4 of the *Kourovka Notebook* [2] reads:

> It is known that if M is a variety (quasivariety, pseudovariety) of groups, then the class IM of all quasigroups that are isotopic to groups in M is also a variety (quasivariety, pseudovariety), and IM is finitely based if and only if M is finitely based. Is it true that if M is generated by a single finite group then IM is generated by a single finite quasigroup? A. A. Gvaramiya

All carriers are nonempty. Quasigroups have multiplication and its two divisions, in the ordinary single-sorted signature. Generation is ordinary generation of the specified class. Here a *pseudovariety* is a class defined by universally quantified finite disjunctions of equations. This is the convention of Gvaramiya's original formulation [1, pp. 1048, 1051]; it must be distinguished from finite closure under homomorphic images, subalgebras and products. An isotopy from a quasigroup to a group consists of three bijections α, β, γ satisfying $\gamma(x * y) = \alpha(x)\beta(y)$.

Theorem 1. *Let G be a finite group.*

1. *If $G \neq 1$ and M is the variety or the quasivariety generated by G, then IM is not generated by a finite quasigroup in the same sense.*
2. *If M is the pseudovariety generated by C_3 in the disjunctive identity sense, then IM is not generated by a finite quasigroup.*
3. *If $G = 1$, the one-element quasigroup generates IM in all three senses.*

The original terminology is the only external mathematical input. The finite permutation identity and finite separation argument below are proved directly. The contribution under review is their application to the finite-generator question; no exact prior full answer was located in the recorded literature search. This search does not certify originality.

2. Varieties and quasivarieties

Define binary terms by

$$T_0(x, y) = y, \qquad T_{k+1}(x, y) = x * T_k(x, y) \quad (k \geq 0).$$

If F is a quasigroup with q elements and $N = q!$, then

$$T_N(x, y) = y \tag{1}$$

is an identity of F. Indeed, every left translation $L_x : y \mapsto x * y$ is a permutation of q elements, so $L_x^N = \mathrm{id}$; induction identifies $T_N(x, y)$ with $L_x^N(y)$. The class defined by (1) is both a variety and a quasivariety. Consequently the variety and the quasivariety generated by F satisfy this identity.

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Now fix $G \neq 1$, choose $g \in G \setminus \{1\}$, and put $m = N + 1$. On $P = G^m$, index coordinates by $\mathbb{Z}/m\mathbb{Z}$ and define the cyclic shift σ by $(\sigma(x))_i = x_{i-1}$. Define

$$x * y = x\sigma(y), \qquad x\backslash z = \sigma^{-1}(x^{-1}z), \qquad z/y = z\sigma(y)^{-1}, \tag{2}$$

where the operations on the right are those of the group P. Substitution gives

$$x\backslash(x * y) = y, \quad x * (x\backslash z) = z, \quad (x * y)/y = x, \quad (z/y) * y = z.$$

Thus (2) defines a quasigroup, and $(\mathrm{id}, \sigma, \mathrm{id})$ is an isotopy to P.

The left translation by 1_P is σ. Let $v \in P$ have entry g at coordinate zero and entry 1 elsewhere. The unique nonidentity entry of $\sigma^N(v)$ is at coordinate $N \not\equiv 0 \pmod m$. Hence

$$T_N(1_P, v) = \sigma^N(v) \neq v.$$

Finite direct powers of G belong to both the variety and the quasivariety it generates, so this quasigroup belongs to IM in either case. If IM were generated by a finite F, choosing $N = |F|!$ would contradict (1). This proves part (1) of Theorem 1.

3. Disjunctive identities

For $n \geq 1$, the universal disjunction

$$\bigvee_{0 \leq i < j \leq n} x_i = x_j \tag{3}$$

holds exactly on carriers of size at most n. Therefore every member of the pseudovariety generated by an n-element algebra has at most n elements.

Lemma 2. *Let A, B be finite quasigroups with $|A| \leq |B|$. If B belongs to the pseudovariety generated by A, then $A \cong B$.*

Proof. The case $|B| = 1$ follows from nonemptiness. Otherwise enumerate $B = \{b_0, \ldots, b_{n-1}\}$ and write $b_i * b_j = b_{\mu(i,j)}$. Let D_B be the universal disjunction of the equations

$$\begin{aligned} x_i &= x_j && (0 \leq i < j < n), \\ x_i * x_j &= x_k && (0 \leq i, j, k < n,\ k \neq \mu(i,j)). \end{aligned}$$

The tuple $(b_0, \ldots, b_{n-1})$ falsifies D_B in B. Suppose $A \not\cong B$ and take any n-tuple in A. A repeated entry satisfies the first kind of disjunct. Otherwise the tuple enumerates A, since $|A| \leq n$. If every disjunct of the second kind failed, all its multiplication entries would have to equal their B-prescribed outputs. The enumeration would give a multiplicative isomorphism $B \to A$, which also preserves the uniquely determined divisions. This is impossible. Thus D_B holds in A and throughout its generated pseudovariety, contradicting its failure in B. □

Take M to be the pseudovariety generated by C_3. By (3), every member of M, and hence of IM, has at most three elements. Besides C_3 itself, IM contains the quasigroup on $\mathbb{Z}/3\mathbb{Z}$ defined by

$$x * y = x - y, \qquad x\backslash z = x - z, \qquad z/y = z + y.$$

The four quasigroup laws follow by substitution, and $(\mathrm{id}, -\mathrm{id}, \mathrm{id})$ is an isotopy to the additive group C_3. This quasigroup is not associative:

$$(0 * 1) * 1 = 1, \qquad 0 * (1 * 1) = 0.$$

If a finite quasigroup F generated IM, then $F \in IM$ and $|F| \leq 3$. Since $C_3 \in IM$, Lemma 2 would imply $F \cong C_3$. Associativity, being a single equation, would then hold throughout the pseudovariety generated by F. The displayed isotope contradicts this, proving part (2) of Theorem 1.

Finally, the identity $x = y$ holds in the trivial group and in each of the three classes it generates. All its isotopes are one-element quasigroups. This proves part (3).

Remark. Under the alternate finite-HSP convention for pseudovarieties, part (1) also holds for every nontrivial finite G. The identity (1) is preserved by homomorphic images, subalgebras and finite products, and the witness (2) is finite. These preservation statements follow directly from term evaluation under homomorphisms, restrictions and coordinatewise products. This observation concerns a different convention from part (2).

Open questions

The original three questions have the negative answers above. For the disjunctive-identity convention, the argument singles out C_3; it does not classify the finite groups for which a single finite quasigroup generates the isotopic closure.

Methods and AI disclosure

This draft was assembled with `gpt-6-astra` in Codex from an internal proof candidate. The original argument passed two independent adversarial reviews, and a separate source audit checked the original terminology against the retained primary pages. Independent Python and GAP calculations corroborated finite instances; no computation is a premise of the proof. The source texts, review reports and transcripts are preserved in the accompanying repository under `problems/9.4/`. This draft is an internal candidate awaiting human assessment; it does not assert publication, editor acceptance or certified originality.

References

1. A. A. Gvaramiya, *Quasigroup classes that are invariant under isotopy, and abstract classes of invertible automata*, Dokl. Akad. Nauk SSSR **282** (1985), no. 5, 1047–1051, Russian.
2. E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory: The Kourovka notebook*, 21 ed., Sobolev Institute of Mathematics, Novosibirsk, 2026, Version arXiv:1401.0300v45, 3 July 2026, Problem 9.4.